The Basics of Complex Numbers

Activity Group Members:

Directions: Work in groups of two to four students on this activity. Each group will submit one solution consisting of a copy of this handout with solutions neatly handwritten in the space provided. This writeup should adhere to the *Guidelines for Written Assignments* documents posted at the course web site. (Extra copies of this handout are also available there.) You may seek assistance on this activity only from other members of your group and from your instructor.

Table 1: Sandwich nutritional data

Sandwich	Turkey Breast	Club	Veggie Sub	Breakfast Sandwich
Calories	280	320	230	470
Fat (gm.)	4.5	6	3	19
Cholesterol (gm.)	20	35	0	200
Carbohydrates (gm.)	46	47	44	53
Fiber (gm.)	5	5	5	5
Protein (gm.)	18	24	9	28
Vitamin A (pct. RDA)	8	8	8	10
Vitamin C (pct. RDA)	35	35	35	15
Calcium (pct. RDA)	6	8	6	25
Iron (pct. RDA)	25	30	25	25

Here are examples involving Maple exponentiation

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> n:=2:

> x:=7:

> x^n;
                                  49
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All of us have encountered complex numbers at one time or another. For most of us, the first time occurs when solving quadratic equations. As an example, suppose that $x^2 - 6x + 13 = 0$. Then, by the quadratic formula,

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} \\ &= \frac{6 \pm \sqrt{-16}}{2} \\ &= \frac{6 \pm 4\sqrt{-1}}{2} \\ &= 3 \pm 2i. \end{aligned}$$

Typically in the past, we have ignored such answers, deeming them “second class citizens,” so to speak, as numbers go. However, in this course, complex numbers *reign supreme*!

A complex number is any number of the form $a+bi$, where $i = \sqrt{1}$, and a and b are real numbers. Of course, $i^2 = -1$. Typically, we use letters like z and w to denote complex numbers, e.g.,

$$z = 3 + 2i.$$

If $z = a + bi$, then b is called the *imaginary part* of z, denoted $\mathfrak{Im}(z)$, and a is called the *real part*, denoted $\mathfrak{Re}(z)$. The *complex conjugate* of a complex number, z, is defined as $\overline{z} = a - bi$. Thus for $z = 3 + 2i$, $\mathfrak{Re}(z) = 3$, $\mathfrak{Im}(z) = 2$, and $\overline{z} = 3 - 2i$.

Of course, every real number may be viewed as a complex number having imaginary part equal to zero. The set of complex numbers is denoted by $\mathbb{C}$. In terms of sets of numbers,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C},$$

where $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of natural numbers, integers, rational numbers, and real numbers, respectively. Complex numbers are added together by adding their respective real and imaginary parts. For example, if $z = 3+2i$ and $w = -1+5i$, then $z+w = 2+7i$. Complex numbers be multiplied by real scalars, e.g., $4z = 4(3 + 2i) = 12 + 8i$, and can also be multiplied by one another using a distributive property for multiplication:

$$\begin{aligned} z \cdot w &= (3 + 2i)(-1 + 5i) \\ &= -3 + 15i - 2i + 10i^2 \\ &= -3 + 13i - 10 \\ &= -13 + 13i. \end{aligned}$$

1. Assume $z = 3 + 2i$ and $w = -1 + 5i$. Express each of the following quantities in the form $a + bi$.

   (a) $\mathfrak{Re}(z) + \mathfrak{Re}(w)$

   (b) $\mathfrak{Re}(z + w)$

   (c) $2z + 3w$

   (d) $z(2 - i)$

   (e) $\overline{z} + \overline{w}$

   (f) $\overline{z + w}$

   (g) $\overline{zw}$

   (h) $\dfrac{1}{z}$ (Hint: Multiply the numerator and denominator by $\overline{z}$.)

(i) $\overline{\left(\dfrac{z}{w+z}\right)}$

2. Assume $z = a+bi$ and $w = c+di$, where a,b,c, and d belong to $\mathbb{R}$. For each statement below, determine whether the statement is true or false. Justify your reasoning with a brief proof or a counterexample. (By a proof, we mean we start with one side of the equation and manipulate it to obtain the other side of the equation. We do not work with both sides at the same time.[1])

   (a) $\overline{zw} = \overline{z} \cdot \overline{w}$

   (b) $\mathfrak{Re}(zw) = \mathfrak{Re}(z) \cdot \mathfrak{Re}(w)$

Because the complex numbers can be added and multiplied together in the ways one would expect and because of other basic properties, they form what students who have completed MTH 310 know to be a *field*. Moreover, if we view scalars as real numbers, the set of complex numbers is a *vector space*.

[1]Students who have completed MTH 210 should be familiar with this idea.

That $\mathbb{C}$ forms a vector space gives us a graphical perspective of complex numbers. This is accomplished by means of what we refer to as the *complex plane*. We identify the horizonal axis with the real part of the complex number and the vertical axis with the imaginary part. A complex number $z = a + bi$ is then plotted as the point (a, b) shown in Figure **??**.

Using this vector representation of a complex number, we can speak of its "length." The *modulus* or *absolute value* or *norm* of a complex number $z = a+bi$ is its distance to the origin in the complex plane. We use the notation $|z|$ to denote the modulus of z. By the Pythagorean Theorem, $|z| = \sqrt{a^2 + b^2}$. For example, if $z = 1 + 2i$, then $|z| = \sqrt{1^2 + 2^2} = \sqrt{5}$.

3. Assume that $z = 1 + 2i$ and $w = -2 + 3i$. Plot each of these Figure **??**. Then calculate each quantity below and graph the result in Figure **??**.

   (a) $z + w$. (Observe how the result compares with the "parallelogram" or "tip-to-tail" method for adding vectors.)

   (b) $\overline{z}$

   (c) $-w$.

   (d) $\mathfrak{Im}(z)$

4. Based upon the preceding examples and other similar ones you may wish to consider, explain the graphical relationship that exists between z and $\overline{z}$. Then do the same for z and $-z$. For each pair of numbers, describe the relationship in terms of the concept of *reflections*.

5. Show that $|z + w| \neq |z| + |w|$.

6. Suppose $z = a + bi$. Show that $z \cdot \overline{z} = |z|^2$.

The modulus provides a convenient means of describing certain sets in the complex plane. For example, the set of numbers $S = \{z \mid z \in \mathbb{C} \text{ and } |z| = 1\}$, which we can abbreviate simply as $|z| = 1$, is the set of all complex numbers that are one unit away from the origin in the complex plane. This of course is really nothing more than the unit circle. The set $|z| < 1$ is the set of complex numbers less than one unit away from this origin, a set we refer to as the *open unit disk*. The word "open" emphasizes the fact that the boundary, i.e., the unit circle itself, is not included. Graphically, the open unit disk is shown in Figure **??**.

Disks of different radii and centered at other complex numbers can be expressed using similar notation. For example, the set $|z - (1 + 2i)| < 3$ consists of the open disk of radius 3 having center at the point $z_0 = 1 + 2i$.

7. Read Example 2 on p. 9 of the text. Then sketch each of the following sets in the complex plane. Support each answer with appropriate calculations and/or a brief explanation.

   (a) $|z + 1 + 2i| < 2$

(b) $|z| \geq 2$

(c) $\mathfrak{Im}(z+i) < 3$

(d) $|z-1|^2 + |z-i|^2 = 2$