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Instructors Solutions Manual for Linear and Nonlinear Programming with Maple: An Interactive, Applications-Based Approach

Contents

Part I

Linear Programming

Chapter 1

An Introduction to Linear Programming

1.1 The Basic Linear Programming Problem Formulation

1. Express each LP below in matrix inequality form. Then solve the LP using Maple provided it is feasible and bounded.

   (a)

$$\begin{aligned} \text{maximize } z &= 6x_1 + 4x_2 \\ \text{subject to} \\ 2x_1 + 3x_2 &\leq 9 \\ x_1 \quad\quad &\geq 4 \\ x_2 &\leq 6 \\ x_1, x_2 &\geq 0, \end{aligned}$$

   The second constraint may be rewritten as $-x_1 \leq -4$ so that matrix inequality form of the LP is given by

$$\begin{aligned} \text{maximize } z &= \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}, \end{aligned} \tag{1.1}$$

   where $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 6 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 9 \\ -4 \\ 6 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

   The solution is given by $\mathbf{x} = \begin{bmatrix} 4.5 \\ 0 \end{bmatrix}$, with $z = 27$.

   (b)

$$\begin{aligned} \text{maximize } z &= 3x_1 + 2x_2 \\ \text{subject to} \\ x_1 \quad\quad &\leq 4 \\ x_1 + 3x_2 &\leq 15 \\ 2x_1 + x_2 &= 10 \\ x_1, x_2 &\geq 0. \end{aligned}$$

   The third constraint can be replaced by the two constraints, $2x_1 + x_2 \leq 10$ and $-2x_1 - x_2 \leq -10$. Thus, the matrix inequality form of the LP is (1.1), with $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 1 \\ -2 & -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 15 \\ 10 \\ -10 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The solution is given by $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, with $z = 17$.

(c)

$$\begin{aligned} &\text{maximize } z = -x_1 + 4x_2 \\ &\text{subject to} \\ &-x_1 + x_2 \le 1 \\ &x_1 + \quad \le 3 \\ &x_1 + x_2 \ge 5 \\ &x_1, x_2 \ge 0. \end{aligned}$$

The third constraint is identical to $-x_1 - x_2 \le -5$. The matrix inequality form of the LP is (1.1), with $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -1 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The solution is $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, with $z = 13$.

(d)

$$\begin{aligned} &\text{minimize } z = -x_1 + 4x_2 \\ &\text{subject to} \\ &x_1 + 3x_2 \ge 5 \\ &x_1 + x_2 \ge 4 \\ &x_1 - x_2 \le 2 \\ &x_1, x_2 \ge 0. \end{aligned}$$

To express the LP in matrix inequality form with the goal of maximization, we set

$$\mathbf{c} = -\begin{bmatrix} -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -4 \end{bmatrix}.$$

The first two constraints may be rewritten as $-x_1 - 3x_2 \le -5$ and

$-x_1 - x_2 \le -4$. The matrix inequality form of the LP becomes (1.1), with $A = \begin{bmatrix} -1 & -3 \\ -1 & -1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ -4 \\ 2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The solution is given by $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, with $z = -1$.

(e)

$$\begin{aligned} \text{maximize } z = 2x_1 - x_2 \\ \text{subject to} \\ x_1 + 3x_2 \ge 8 \\ x_1 + x_2 \ge 4 \\ x_1 - x_2 \le 2 \\ x_1, x_2 \ge 0. \end{aligned}$$

The first and second constraints are identical to $-x_1 - 3x_2 \le -8$ and $-x_1 - x_2 \le -4$, respectively. Thus, $A = \begin{bmatrix} -1 & -3 \\ -1 & -1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -8 \\ -4 \\ 2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The LP is unbounded.

(f)

$$\begin{aligned} \text{minimize } z = 2x_1 + 3x_2 \\ \text{subject to} \\ 3x_1 + x_2 \ge 1 \\ x_1 + x_2 \le 6 \\ x_2 \ge 0. \end{aligned}$$

Define $x_1 = x_{1,+} - x_{1,-}$, where $x_{1,+}$ and $x_{1,-}$ are nonnegative. Then, as a maximization problem, the LP may be rewritten in terms of three decision variables as

$$\begin{aligned} \text{maximize } z = -2(x_{1,+} - x_{1,-}) - 3x_2 \\ \text{subject to} \\ -3(x_{1,+} - x_{1,-}) - x_2 \le -1 \\ (x_{1,+} - x_{1,-}) + x_2 \le 6 \\ x_{1,+}, x_{1,-}, x_2 \ge 0. \end{aligned}$$

The matrix inequality form of the LP becomes (1.1), with $A = \begin{bmatrix} -3 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -2 & 2 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_{1,+} \\ x_{1,-} \\ x_2 \end{bmatrix}$.

The solution is given by $x_{1,+} = \frac{1}{3}$ and $x_{1,-} = x_2 = 0$, with $z = -\frac{2}{3}$.

2. The LP is given by

$$\begin{aligned} &\text{minimize } z = x_1 + 4x_2 \\ &\text{subject to} \\ &x_1 + 2x_2 \le 5 \\ &|x_1 - x_2| \le 2 \\ &x_1, x_2 \ge 0. \end{aligned}$$

The constraint involving the absolute value is identical to $-2 \le x_1 - x_2 \le 2$, which may be written as the two constraints, $x_1 - x_2 \le 2$ and $-x_1 + x_2 \le 2$. The matrix inequality form of the LP is (1.1) with $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -1 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The solution is given by $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, with $z = 0$.

3. If x_1 and x_2 denote the number of chairs and number of tables, respectively produced by the company, then $z = 5x_1 + 7x_2$ denotes the revenue, which we seek to maximize. The number of square blocks needed to produce x_1 chairs is $2x_1$, and the number of square blocks needed to produce x_2 tables is $2x_2$. Since six square blocks are available, we have the constraint, $2x_1 + 2x_2 \le 6$. Similar reasoning, applied to the rectangular blocks, leads to the constraint $x_1 + 2x_2 \le 8$. Along with sign conditions, these results yield the LP

$$\begin{aligned} &\text{maximize } z = 5x_1 + 7x_2 \\ &\text{subject to} \\ &2x_1 + 2x_2 \le 6 \\ &x_1 + 2x_2 \le 8 \\ &x_1, x_2 \ge 0. \end{aligned}$$

.

For the matrix inequality form, $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 5 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The solution is given by $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, with $z = 21$.

4. (a) If x_1 and x_2 denote the number of grams of grass and number of grams of forb, respectively, consumed by the vole on a given day, then the total foraging time is given by $z = 45.55x_1 + 21.87x_2$. Since the coefficients in this sum have units of minutes per gram, the units of z are minutes.

   The products $1.64x_1$ and $2.67x_2$ represent the amount of digestive capacity corresponding to eating x_1 grams of grass and x_2 grams of forb, respectively. The total digestive capacity is 31.2 gm-wet, which yields the constraint $1.64x_1 + 2.67x_2 \leq 31.2$. Observe that the units of the variables, x_1 and x_2, are gm-dry. Each coefficient in this inequality has units of gm-wet per gm-dry, so the sum $1.64x_1 + 2.67x_2$ has the desired units of gm-wet.

   Similar reasoning focusing on energy requirements, leads to the constraint $2.11x_1 + 2.3x_2 \geq 13.9$. Along with sign conditions, we arrive at

$$\begin{aligned} \text{minimize } z = 45.55x_1 + 21.87x_2 \\ \text{subject to} \\ 1.64x_1 + 2.67x_2 \leq 31.2 \\ 2.11x_1 + 2.3x_2 \geq 13.9 \\ x_1, x_2 \geq 0. \end{aligned}$$

   (b) The solution, obtained using Maple's `LPSolve` command, is given by $x_1 = 0$ grams of grass and $x_2 \approx 6.04$ grams of forb. The total time spent foraging is $z \approx 132.17$ minutes.

Solutions to Waypoints

Waypoint 1.1.1. There are of course many possible combinations. Table 1.1 summarizes the outcomes for four choices.

The fourth combination requires 35 gallons of stock B, so it does not satisfy the

TABLE 1.1: Production combinations (in gallons)

Premium	Reg. Unleaded	Profit ($)
5	5	3.5
5	7	4.1
7	5	4.3
7	7	4.9

listed constraints. Of the three that do, the combination of 7 gallons premium and 5 gallons regular unleaded results in the greatest profit.

1.2 Linear Programming: A Graphical Perspective in $\mathbb{R}^2$

1. (a)

$$\begin{aligned} \text{Maximize } z &= 6x_1 + 4x_2 \\ \text{subject to} & \\ 2x_1 + 3x_2 &\le 9 \\ x_1 \quad &\ge 4 \\ x_2 &\le 6 \\ x_1, x_2 &\ge 0, \end{aligned}$$

The feasible region is shown in Figure 2.2. The solution is given by $\mathbf{x} = \begin{bmatrix} 4.5 \\ 0 \end{bmatrix}$, with $z = 27$.

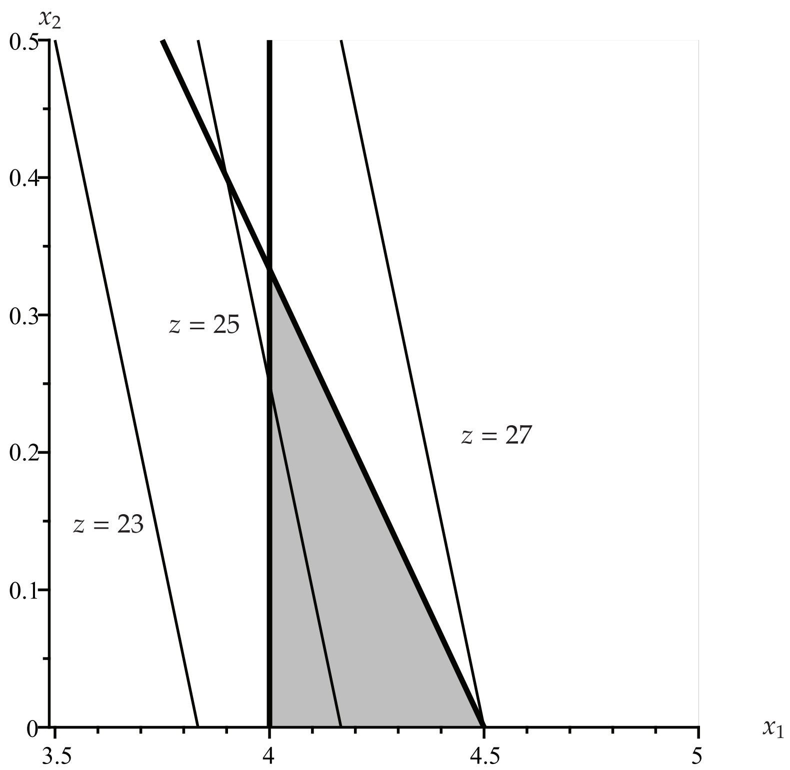

FIGURE 1.1: Feasible region with contours $z = 23$, $z = 25$ and $z = 27$.

(b)

$$\begin{aligned} \text{maximize } z &= 3x_1 + 2x_2 \\ \text{subject to} & \\ x_1 \qquad &\leq 4 \\ x_1 + 3x_2 &\leq 15 \\ 2x_1 + x_2 &= 10 \\ x_1, x_2 &\geq 0. \end{aligned}$$

The feasible region is the line segment shown in Figure 1.2. The solution is given by $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, with $z = 17$.

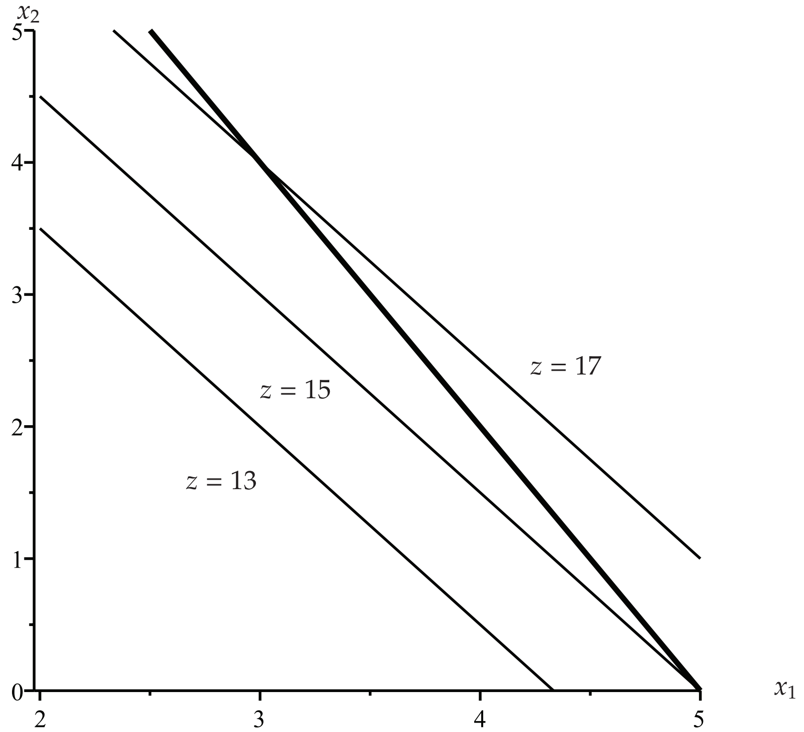

FIGURE 1.2: Feasible region with contours $z = 13$, $z = 15$ and $z = 17$.

(c)

$$\begin{aligned} \text{minimize } z &= -x_1 + 4x_2 \\ \text{subject to} & \\ x_1 + 3x_2 &\geq 5 \\ x_1 + x_2 &\geq 4 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

The feasible region is shown in Figure 1.3. The solution is given by $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, with $z = 1$.

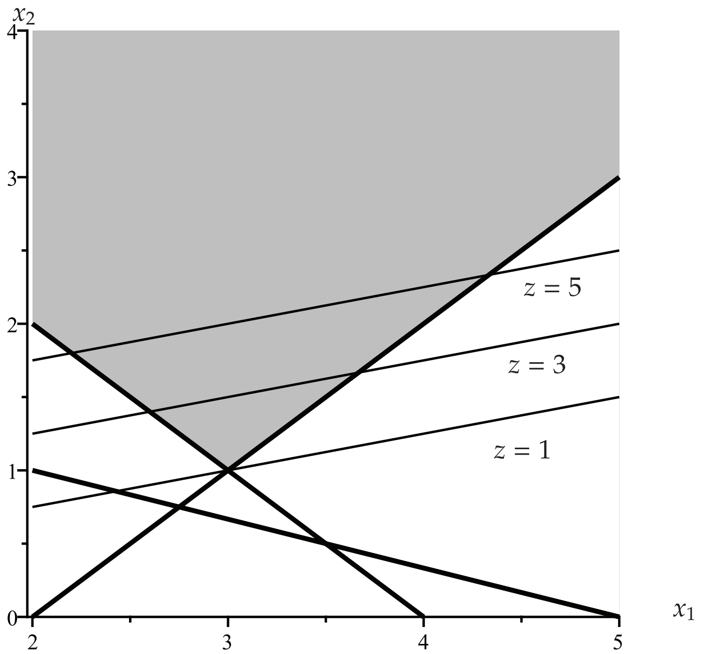

FIGURE 1.3: Feasible region with contours $z = 5$, $z = 3$ and $z = 1$.

(d)

$$\begin{aligned} \text{maximize } z &= 2x_1 + 6x_2 \\ \text{subject to} & \\ x_1 + 3x_2 &\leq 6 \\ x_1 + 2x_2 &\geq 5 \\ x_1, x_2 &\geq 0. \end{aligned}$$

The feasible region is shown in Figure 1.4. The LP has alternative optimal solutions that fall on the segment connecting $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ to $\mathbf{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$. Each such solution has an objective value of $z = 12$, and the parametric representation of the segment is given by

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 3t + 6(1-t) \\ t + 0(1-t) \end{bmatrix} \\ &= \begin{bmatrix} 6-3t \\ t \end{bmatrix}, \end{aligned}$$

where $0 \leq t \leq 1$.

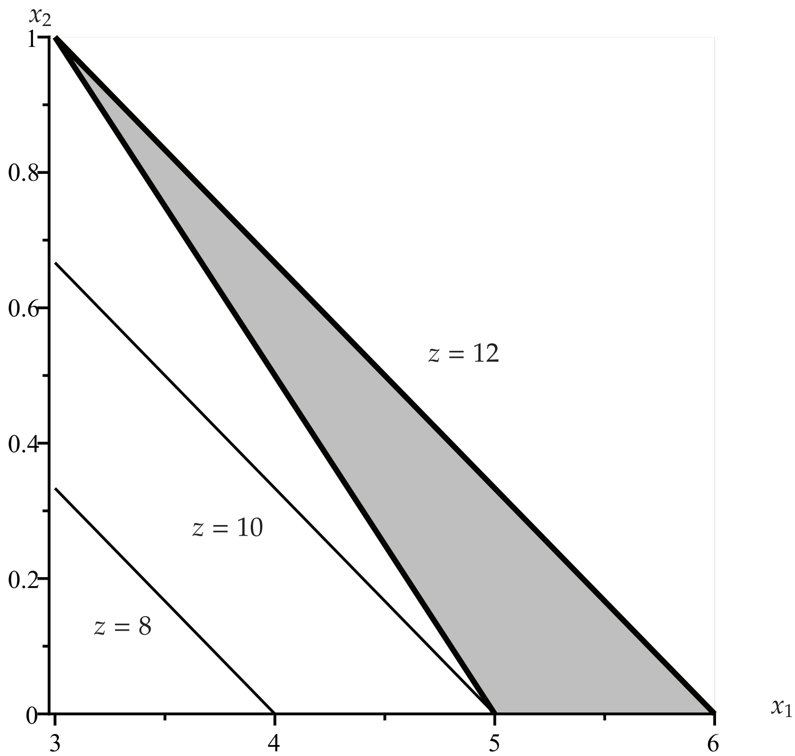

FIGURE 1.4: Feasible region with contours $z = 8$, $z = 10$ and $z = 12$.

(e)

$$\begin{aligned} \text{minimize } z &= 2x_1 + 3x_2 \\ \text{subject to} & \\ 3x_1 + x_2 &\geq 1 \\ x_1 + x_2 &\leq 6 \\ x_2 &\geq 0. \end{aligned}$$

The feasible region is shown in Figure 1.5. The solution is given by $\mathbf{x} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$, with $z = \frac{2}{3}$.

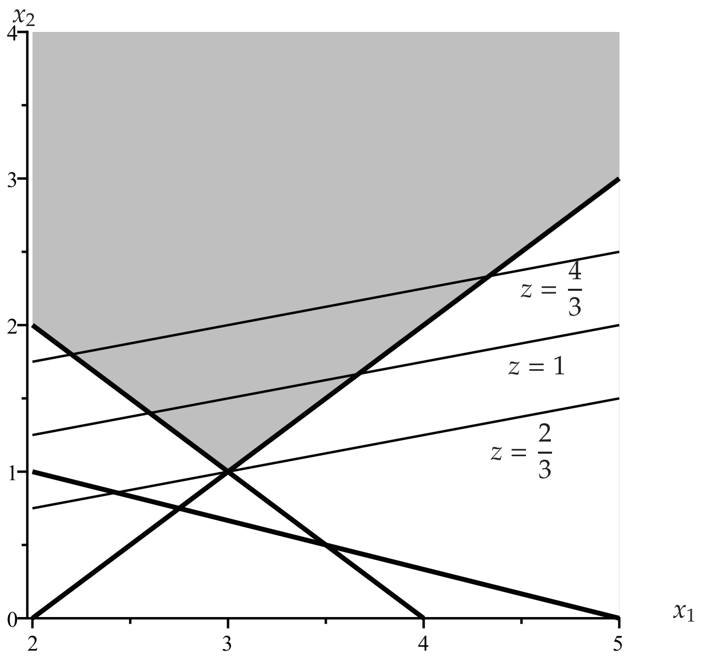

FIGURE 1.5: Feasible region with contours $z = \frac{4}{3}$, $z = 1$ and $z = \frac{2}{3}$.

2. The *Foraging Herbivore Model*), Exercise 4, from Section 1.1 is given by

$$\begin{aligned} \text{minimize } z &= 45.55x_1 + 21.87x_2 \\ \text{subject to} & \\ 1.64x_1 + 2.67x_2 &\leq 31.2 \\ 2.11x_1 + 2.3x_2 &\geq 13.9 \\ x_1, x_2 &\geq 0, \end{aligned}$$

whose feasible region is shown in Figure 1.6. The solution is given by $\mathbf{x} \approx \begin{bmatrix} 0 \\ 6.043 \end{bmatrix}$, with $z \approx 132.171$.

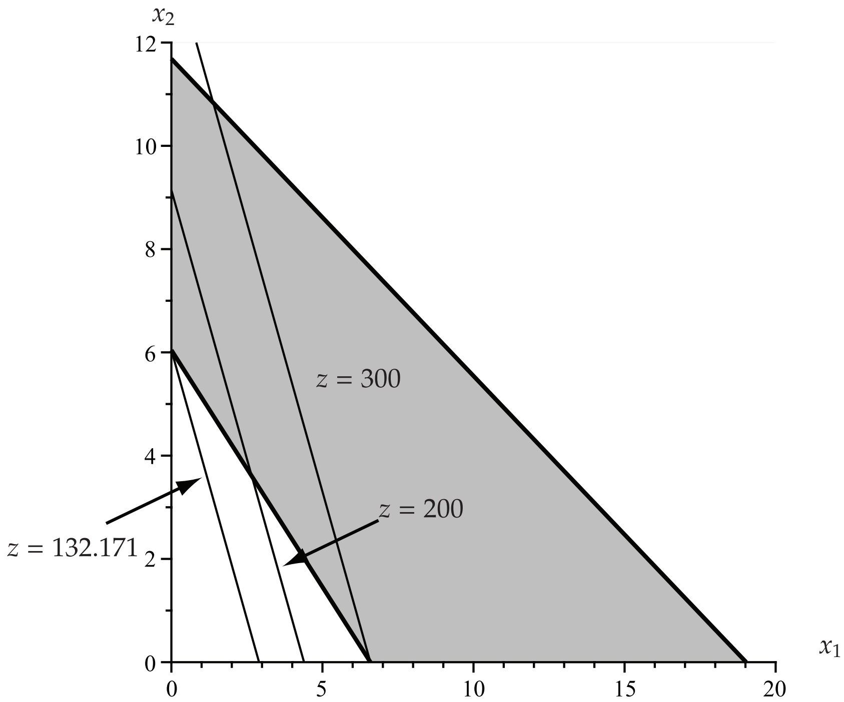

FIGURE 1.6: Feasible region with contours $z = 300$, $z = 200$ and $z = 132.171$.

3. (a) The feasible region, along with the contours $z = 50$, $z = 100$, and $z = 150$, is shown in Figure 1.7

   (b) If M is an arbitrarily large positive real number, then the set of points falling on the contour $z = M$ satisfies $2x_1 - x_2 = M$, or, equivalently, $x_2 = 2x_1 - M$. However, as Figure 1.7 indicates, to use the portion of this line that falls within the feasible region, we must have $x_1 - x_2 \leq 2$. Combining this inequality with $2x_1 - x_2 = M$ yields $x_1 \geq M - 2$. Thus, the portion of the contour $z = M$ belonging to the feasible region consists of the ray,

$$\{(x_1, x_2) \mid x_1 \geq M - 2 \text{ and } x_2 = 2x_1 - M\}.$$

   (c) Fix M and consider the starting point on the ray from (b). Then

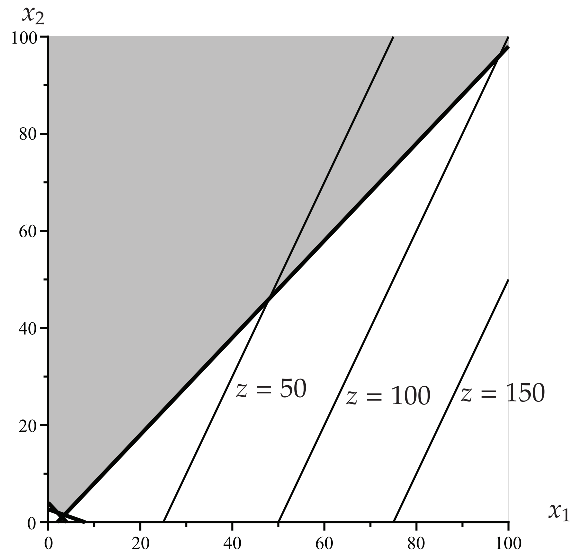

FIGURE 1.7: Feasible region with contours $z = 50$, $z = 100$ and $z = 150$.

$x_1 = M - 2$ and $x_2 = 2x_1 - M = M - 4$, in which case all constraints and sign conditions are satisfied if M is large enough. (Actually, $M \geq 5$ suffices.) Since M can be made as large as we like, the LP is unbounded.

4. Suppose f is a linear function of x_1 and x_2, and consider the LP given by

$$\begin{aligned} \text{maximize } z &= f(x_1, x_2) \\ \text{subject to} & \\ x_1 \quad &\geq 1 \\ x_2 &\leq 1 \\ x_1, x_2 &\geq 0, \end{aligned} \tag{1.2}$$

   (a) The feasible region is shown in Figure 1.8
   (b) If $f(x_1, x_2) = x_2 - x_1$, then $f(1, 1) = 0$. For all other feasible points, the objective value is negative. Hence, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the unique optimal solution.
   (c) If $f(x_1, x_2) = x_1$, then the LP is unbounded.

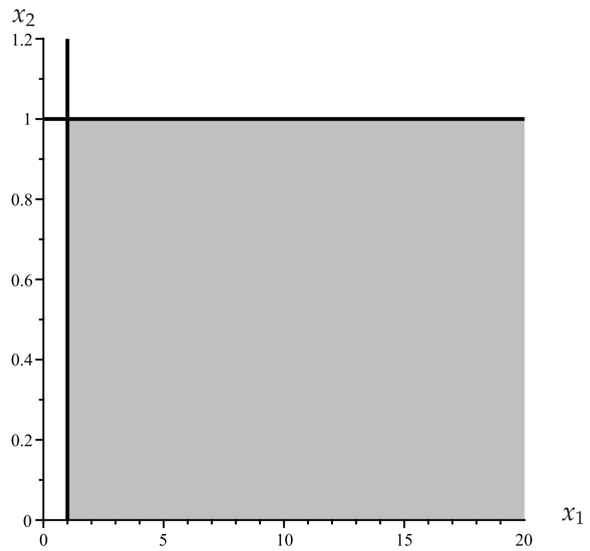

FIGURE 1.8: Feasible region for LP 1.2.

(d) If $f(x_1, x_2) = x_2$, then the LP has alternative optimal solutions.

Solutions to Waypoints

Waypoint 1.2.1. The following Maple commands produce the desired feasible region:

```
> restart:with(plots):
> constraints:=[x1<=8,2*x1+x2<=28,3*x1+2*x2<=32,x1>=0,x2>=0]:
> inequal(constraints,x1=0..10,x2=0..16,optionsfeasible=(color=grey),
  optionsexcluded=(color=white),optionsclosed=(color=black),thickness=1);
```

Waypoint 1.2.2. The following commands produce the feasible region, upon which are superimposed the contours, $z = 20$, $z = 30$, and $z = 40$.

```
> restart:with(plots):
> f:=(x1,x2)->4*x1+3*x2;
```

$$f := (x_1, x_2) \to 4x_1 + 3x_2$$

```
> constraints:=[x1<=8,2*x1+x2<=28,3*x1+2*x2<=32,x1>=0,x2>=0]:
> FeasibleRegion:=inequal(constraints,x1=0..10,x2=0..16,optionsfeasible=(color=grey),
  optionsexcluded=(color=white),optionsclosed=(color=black),thickness=2):
```

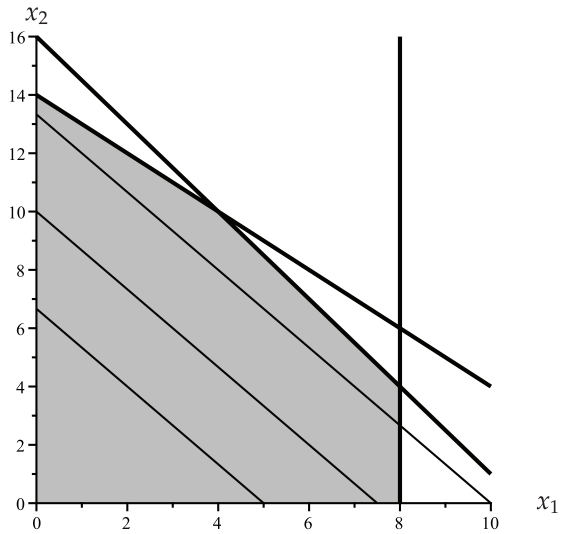

FIGURE 1.9: Feasible region and object contours $z = 20$, $z = 30$, and $z = 40$.

```
> ObjectiveContours:=contourplot(f(x1,x2),x1=0..10,x2=0..10,contours=[20,30,40],thicknes
> display(FeasibleRegion, ObjectiveContours);
```

The resulting graph is shown in Figure 1.9
Maple does not label the contours, but the contours must increase in value as x_1 and x_2 increase. This fact, along with the graph, suggests that the solution is given by $x_1 = 4$, $x_2 = 10$.

1.3 Basic Feasible Solutions

1. For each of the following LPs, write the LP in the standard matrix form

$$\begin{aligned} &\text{maximize } z = \mathbf{c} \cdot \mathbf{x} \\ &\text{subject to} \\ &[A|I_m]\begin{bmatrix}\mathbf{x}\\ \mathbf{s}\end{bmatrix} = \mathbf{b} \\ &\mathbf{x}, \mathbf{s} \geq \mathbf{0}. \end{aligned}$$

Then determine all basic and basic feasible solutions, expressing each solution in vector form. Label each solution next to its corresponding point on the feasible region graph.

(a)

$$\begin{aligned} \text{Maximize } z &= 6x_1 + 4x_2 \\ \text{subject to} & \\ 2x_1 + 3x_2 &\leq 9 \\ x_1 \quad &\geq 4 \\ x_2 &\leq 6 \\ x_1, x_2 &\geq 0, \end{aligned}$$

Note that this LP is the same as that from Exercise 1a of Section 1.1, where $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 6 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 9 \\ -4 \\ 6 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Since there are three constraints, $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$.

In this case, the matrix equation,

$$[A|I_m]\begin{bmatrix}\mathbf{x}\\ \mathbf{s}\end{bmatrix} = \mathbf{b} \tag{1.3}$$

has at most $\binom{5}{2} = 10$ possible basic solutions. Each is obtained by setting two of the five entries of $\begin{bmatrix}\mathbf{x}\\ \mathbf{s}\end{bmatrix}$ to zero and solving (1.3) for the remaining three, provided the system is consistent. Of the ten possible cases to consider, two lead to inconsistent systems. They arise from setting $x_1 = s_2 = 0$ and from setting $x_2 = s_3 = 0$. The eight remaining consistent systems yield the following results:

i. $x_1 = 0, x_2 = 0, s_1 = 9, s_2 = -4, s_3 = 6$ (basic, but not basic feasible solution)

ii. $x_1 = 0, x_2 = 3, s_1 = 0, s_2 = -4, s_3 = 3$ (basic, but not basic feasible solution)

iii. $x_1 = 0, x_2 = 6, s_1 = -9, s_2 = -4, s_3 = 0$ (basic, but not basic feasible solution)

iv. $x_1 = 4.5, x_2 = 0, s_1 = 0, s_2 = .5, s_3 = 6$ (basic feasible solution)

v. $x_1 = 4, x_2 = 0, s_1 = 1, s_2 = 0, s_3 = 6$ (basic feasible solution)

vi. $x_1 = 4, x_2 = 1/3, s_1 = 0, s_2 = 0, s_3 = 17/3$ (basic feasible solution)

vii. $x_1 = -4.5, x_2 = 6, s_1 = 0, s_2 = -8.5, s_3 = 0$ (basic, but not basic feasible solution)

viii. $x_1 = 4, x_2 = 6, s_1 = -17, s_2 = 0, s_3 = 0$ (basic, but not basic feasible solution)

The feasible region is that from Exercise 1a of Section 1.2, and the solution is $x_1 = 4.5$ and $x_2 = 0$ with $z = 27$.

(b)

$$\begin{aligned} \text{minimize } z = -x_1 + 4x_2 \\ \text{subject to} \\ x_1 + 3x_2 \geq 5 \\ x_1 + x_2 \geq 4 \\ x_1 - x_2 \leq 2 \\ x_1, x_2 \geq 0. \end{aligned}$$

Note that this LP is the same as that from Exercise 1d of Section 1.1, where $A = \begin{bmatrix} -1 & -3 \\ -1 & -1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ -4 \\ 2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Since there are three constraints, $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$.

As in the previous exercise, there are ten possible cases to consider. In this situation, all ten systems are consistent. The results are as follows:

i. $x_1 = 0, x_2 = 0, s_1 = -5, s_2 = -4, s_3 = 2$ (basic, but not basic feasible solution)

ii. $x_1 = 0, x_2 = 5/3, s_1 = 0, s_2 = -7/3, s_3 = 11/3$ (basic, but not basic feasible solution)

iii. $x_1 = 0, x_2 = 4, s_1 = 7, s_2 = 0, s_3 = 6$ (basic feasible solution)

iv. $x_1 = 0, x_2 = -2, s_1 = -11, s_2 = -6, s_3 = 0$ (basic, but not basic feasible solution)

v. $x_1 = 5, x_2 = 0, s_1 = 0, s_2 = 1, s_3 = -3$ (basic, but not basic feasible solution)

vi. $x_1 = 4, x_2 = 0, s_1 = -1, s_2 = 0, s_3 = -2$ (basic, but not basic feasible solution)

vii. $x_1 = 2, x_2 = 0, s_1 = -3, s_2 = -2, s_3 = 0$ (basic, but not basic feasible solution)

viii. $x_1 = 7/2, x_2 = 1/2, s_1 = 0, s_2 = 0, s_3 = -1$ (basic, but not basic feasible solution)

ix. $x_1 = 11/4, x_2 = 3/4, s_1 = 0, s_2 = -1/2, s_3 = 0$ (basic, but not basic feasible solution)

x. $x_1 = 3, x_2 = 1, s_1 = 1, s_2 = 0, s_3 = 0$ (basic feasible solution)

The feasible region is that from Exercise 1c of Section 1.2, and the solution is $x_1 = 3$ and $x_2 = 1$ with $z = 1$.

2. The constraint equations of the standard form of the *FuelPro* LP are given by

$$\begin{aligned} x_1 \quad\quad + s_1 \quad\quad\quad &= 8 \\ 2x_1 + 2x_2 \quad\quad + s_2 \quad &= 28 \\ 3x_1 + 2x_2 \quad\quad\quad + s_3 &= 32 \end{aligned} \tag{1.4}$$

If x_1 and s_1 are nonbasic, then both are zero. In this case (1.4) is inconsistent. Observe that if $x_1 = s_1 = 0$, that the coefficient matrix corresponding to (1.4) is given by $M = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, which is not invertible.

3. The LP is given by

$$\begin{aligned} \text{maximize } z = 3x_1 + 2x_2 \\ \text{subject to} \\ x_1 \quad\quad &\leq 4 \\ x_1 + 3x_2 &\leq 15 \\ 2x_1 + x_2 &= 10 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(a) The third constraint is the combination of $2x_1 + x_2 \leq 10$ and $-2x_1 - 2x_2 \leq -10$. Thus, the matrix inequality form of the LP has $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 2 \\ -2 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 15 \\ 10 \\ -10 \end{bmatrix}$.

(b) Since there are four constraints, there are four corresponding slack variables, s_1, s_2, s_3, and s_4, which lead to the matrix equation

$$\begin{aligned} &\text{maximize } z = \mathbf{c} \cdot \mathbf{x} \\ &\text{subject to} \\ &[A|I_m]\begin{bmatrix}\mathbf{x}\\ \mathbf{s}\end{bmatrix} = \mathbf{b} \\ &\mathbf{x}, \mathbf{s} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix}s_1\\s_2\\s_3\\s_4\end{bmatrix}$. If s_3 and s_4 are nonbasic, then both are zero. In this case, the preceding matrix equation becomes

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix}x_1\\x_2\\s_1\\s_2\end{bmatrix} = \mathbf{b}.$$

The solution to this matrix equation yields $x_1 = 4 - t$, $x_2 = 2 + 2t$, $s_1 = t$, and $s_2 = 5-5t$, where t is a free quantity. A simple calculation shows that for x_1, x_2, s_1, and s_2 to all remain nonnegative, it must be the case that $0 \leq t \leq 1$. Note that if $t = 0$, then $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$; if $t = 1$, then $\mathbf{x} = \begin{bmatrix}3\\4\end{bmatrix}$. As t increases from 0 to 1, $\mathbf{x}$ varies along the line segment connecting these two points.

4. The *FuelPro* LP has basic feasible solutions corresponding to the five points

$$\mathbf{x} = \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}8\\0\end{bmatrix}, \begin{bmatrix}8\\4\end{bmatrix}, \begin{bmatrix}4\\10\end{bmatrix}, \text{ and } \begin{bmatrix}0\\14\end{bmatrix}.$$

Suppose we add a constraint that does not change the current feasible region but passes through one of these points. While there are countless such examples, a simple choice is to add $x_2 \leq 14$. Now consider the original *FuelPro* LP, with this new constraint added, and let s_4 denote the new corresponding slack variable. In the original LP, $\mathbf{x} = \begin{bmatrix}0\\14\end{bmatrix}$ corresponded to a basic feasible solution in which the nonbasic variables were x_1 and s_2. In the new LP, any basic solution is obtained by setting two of the six variables, x_1, x_2, s_1, s_2, s_3, and s_4, to zero. If we again choose x_1 and s_2 as nonbasic, then the resulting system of equations yields a basic feasible solution in which s_4 is a basic variable equal to zero. Thus, the LP is degenerate.

5. A subset V of $\mathbb{R}^n$ is said to be *convex* is whenever two points belong to V, so does the line segment connecting them. In other words, $\mathbf{x}_1, \mathbf{x}_2 \in V$, implies that $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in V$ for all $0 \le t \le 1$.

   (a) Suppose that the LP is expressed in matrix inequality form as

   $$\begin{aligned} \text{maximize } z &= \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} & \\ A\mathbf{x} &\le \mathbf{b} \\ \mathbf{x} &\ge \mathbf{0}. \end{aligned} \tag{1.5}$$

   If $\mathbf{x}_1$ and $\mathbf{x}_2$ are feasible then $A\mathbf{x}_1 \le \mathbf{b}$ and $A\mathbf{x}_2 \le \mathbf{b}$. Thus, if $0 \le t \le 1$, then

   $$\begin{aligned} A\left(t\mathbf{x}_1 + (1-t)\mathbf{x}_2\right) &= tA\mathbf{x}_1 + (1-t)\mathbf{x}_2 \\ &\le t\mathbf{b} + (1-t)\mathbf{b} \\ &= \mathbf{b}. \end{aligned}$$

   By similar reasoning, if $\mathbf{x}_1, \mathbf{x}_2 \ge \mathbf{0}$, then $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \ge \mathbf{0}$ whenever $0 \le t \le 1$.

   This shows that if $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the matrix inequality and sign conditions in (1.5), then so does $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \ge \mathbf{0}$ for $0 \le t \le 1$. Hence, the feasible region is convex.

   (b) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of LP (1.5), then both are feasible. By the previous result, $\mathbf{x} = t\mathbf{x}_1 + (1-t)\mathbf{x}_2$ is also feasible if $0 \le t \le 1$. Since, $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of the LP, they have a common objective value $z_0 = \mathbf{c} \cdot \mathbf{x}_1 = \mathbf{c} \cdot \mathbf{x}_2$. Now consider the objective value at $\mathbf{x}$:

   $$\begin{aligned} \mathbf{c} \cdot \mathbf{x} &= \mathbf{c} \cdot (t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \\ &= t\mathbf{c} \cdot \mathbf{x}_1 + (1-t)\mathbf{c} \cdot \mathbf{x}_2 \\ &= tz_0 + (1-t)z_0 \\ &= z_0. \end{aligned}$$

   Thus, $\mathbf{x} = t\mathbf{x}_1 + (1-t)\mathbf{x}_2$ is also yields an objective value of z_0. Since $\mathbf{x}$ is also feasible, it must be optimal as well.

6. Suppose the matrix inequality form of the LP is given by

   $$\begin{aligned} \text{maximize } z &= \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} & \\ A\mathbf{x} &\le \mathbf{b} \\ \mathbf{x} &\ge \mathbf{0}, \end{aligned} \tag{1.6}$$

and let $\tilde{\mathbf{x}}$ denote an optimal solution. Since $\tilde{\mathbf{x}}$ belongs to the convex hull generated by the set of basic feasible solutions,

$$\tilde{\mathbf{x}} = \sum_{i=1}^{p} \sigma_i \mathbf{x}_i,$$

where the weights σ_i satisfy $\sigma_i \geq 0$ for all i and $\sum_{i=1}^{p} \sigma_i = 1$ and where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ are the basic feasible solutions of (1.6). Relabeling the basic feasible solutions if necessary, we may assume that $\mathbf{c} \cdot \mathbf{x}_i \leq \mathbf{c} \cdot \mathbf{x}_1$, for $1 \leq i \leq p$. Now consider the objective value corresponding to $\tilde{\mathbf{x}}$. We have

$$\begin{aligned}
\mathbf{c} \cdot \tilde{\mathbf{x}} &= \mathbf{c} \cdot \left(\sum_{i=1}^{p} \sigma_i \mathbf{x}_i \right) \\
&= \sum_{i=1}^{p} \mathbf{c} \cdot (\sigma_i \mathbf{x}_i) \\
&= \sum_{i=1}^{p} \sigma_i \left(\mathbf{c} \cdot \mathbf{x}_i \right) \\
&\leq \sum_{i=1}^{p} \sigma_i \left(\mathbf{c} \cdot \mathbf{x}_1 \right) \\
&= \left(\mathbf{c} \cdot \mathbf{x}_1 \right) \sum_{i=1}^{p} \sigma_i \\
&= \mathbf{c} \cdot \mathbf{x}_1.
\end{aligned}$$

Thus the objective value corresponding to $\mathbf{x}_1$ is at least as large as that corresponding to the optimal solution, $\tilde{\mathbf{x}}$, which implies that $\mathbf{x}_1$ is also a solution of (1.6)

Solutions to Waypoints

Waypoint 1.3.1. The matrix equation

$$[A|I_3] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}$$

has a solution given by $\mathbf{x} = \mathbf{0}$ and $\mathbf{s} = \mathbf{b}$. Since the corresponding system has more variables then equations, it must have infinitely many solutions.

Waypoint 1.3.2. To solve the given matrix equation, we row reduce the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 8 \\ 2 & 2 & 0 & 1 & 0 & 28 \\ 3 & 2 & 0 & 0 & 1 & 32 \end{array}\right].$$

The result is given by

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 4 \\ 0 & 1 & 0 & 3/2 & -1 & 10 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right].$$

If we let s_2 and s_3 denote the free variables, then we may write $s_2 = t$, $s_3 = s$, where $t, s \in \mathbb{R}$. With this notation, $x_1 = 4 + t - s$, $x_2 = 10 - \frac{3}{2}t + s$, and $s_1 = 4 - t + s$. We obtain the five extreme points as follows:

1. $t = 28$ and $s = 32$ yield $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 8 \\ 28 \\ 32 \end{bmatrix}$

2. $t = 12$ and $s = 8$ yield $\mathbf{x} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 0 \\ 12 \\ 8 \end{bmatrix}$

3. $t = 4$ and $s = 0$ yield $\mathbf{x} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

4. $t = s = 0$ yields $\mathbf{x} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

5. $t = 0$ and $s = 4$ yield $\mathbf{x} = \begin{bmatrix} 0 \\ 14 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

These extreme points may then be labeled on the feasible region.

Waypoint 1.3.3.

1. The feasible region and various contours are shown in Figure **??**

   The extreme points are given by the $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 6/7 \\ 12/7 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

2. If $\mathbf{c} = [-5, 2]$, $A = \begin{bmatrix} 3 & 2 \\ 8 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, then the standard matrix form is given by

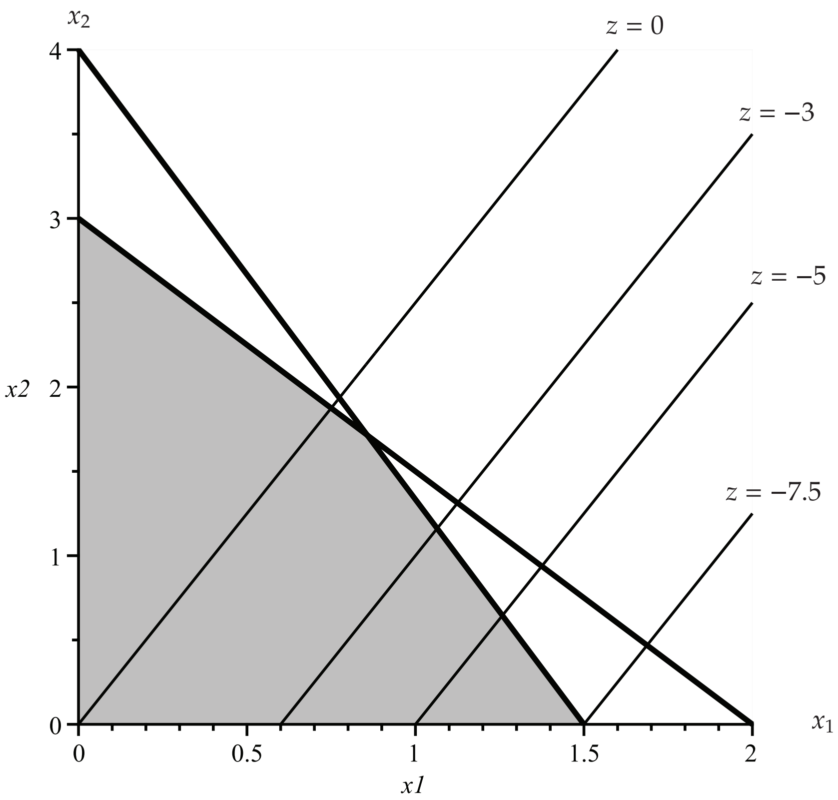

FIGURE 1.10: Feasible region with contours $z = 0$, $z = -3$, $z = -5$ and $z = -7.5$.

$$\begin{aligned} \text{maximize } z &= \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} & \\ [A|I_2]\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} &= \mathbf{b} \\ \mathbf{x}, \mathbf{s} &\geq \mathbf{0}, \end{aligned}$$

3. The two constraint equations can be expressed as $3x_1 + 2x_2 = 6$ and $8x_1 + 3x_2 = 12$. Substituting $x_1 = x_2 = 0$ into this equations yields the basic solution $x_1 = 0$, $x_2 = 0$, $s_1 = 6$ and $s_2 = 16$. Performing the same operations using the other three extreme points from (1) yields the remaining three basic feasible solutions:

   (a) $x_1 = 1.5, x_2 = 0, s_1 = 1.5$ and $s_2 = 0$

(b) $x_1 = \frac{6}{7}$, $x_2 = \frac{12}{7}$, $s_1 = 0$ and $s_2 = 0$

(c) $x_1 = 0$, $x_2 = 3$, $s_1 = 0$ and $s_2 = 16$

4. The contours in Figure **??** indicate the optimal solution occurs at the basic feasible solution $\mathbf{x} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$, where $z = -7.5$.

Chapter 2

The Simplex Algorithm

2.1 The Simplex Algorithm

1. For each LP, we indicate the tableau after each iteration and highlight the pivot entry used to perform the next iteration.

(a)

TABLE 2.1: Initial tableau

z	x_1	x_2	s_1	s_2	s_3	RHS
1	-3	-2	0	0	0	0
0	[1]	0	1	0	0	4
0	1	3	0	1	0	15
0	2	1	0	0	1	10

TABLE 2.2: Tableau after first iteration

z	x_1	x_2	s_1	s_2	s_3	RHS
1	0	-2	3	0	0	12
0	1	0	1	0	0	4
0	0	3	-1	1	0	11
0	0	[1]	-2	0	1	2

TABLE 2.3: Tableau after second iteration

z	x_1	x_2	s_1	s_2	s_3	RHS
1	0	0	-1	0	2	16
0	1	0	1	0	0	4
0	0	0	[5]	1	-3	5
0	0	1	-2	0	1	2

The solution is $x_1 = 3$ and $x_2 = 4$, with $z = 17$.

(b)

TABLE 2.4: Final tableau

z	x_1	x_2	s_1	s_2	s_3	RHS
1	0	0	0	1/5	7/5	17
0	1	0	0	-1/5	3/5	3
0	0	0	1	1/5	-3/5	1
0	0	1	0	2/5	-1/5	4

TABLE 2.5: Initial tableau

z	x_1	x_2	s_1	s_2	s_3	RHS
1	-4	-3	0	0	0	0
0	1	0	1	0	0	4
0	-2	1	0	1	0	12
0	1	2	0	0	1	14

The solution is $x_1 = 4$ and $x_2 = 5$, with $z = 31$.

(c)

The solution is $x_1 = 2$, $x_2 = 0$, and $x_3 = \frac{10}{3}$, with $z = \frac{74}{3}$.

(d)

This is a minimization problem, so we terminate the algorithm when all coefficients in the top row that correspond to nonbasic variables are nonpositive. The solution is $x_1 = 0$ and $x_2 = 4$, with $z = -8$.

2. If we rewrite the first and second constraints as $-x_1 - 3x_2 \leq -8$ and $-x_1 - x_2 \leq -4$ and incorporate slack variables as usual, our initial tableau becomes that shown in Table 2.13

   The difficulty we face when attempting to perform the simplex algorithm, centers on the fact that the origin, $(x_1, x_2) = (0, 0)$ yields negative values of s_1 and $_2$. In other words, the origin corresponds to a basic, but not basic feasible, solution. We cannot read off the initial basic feasible solution as we could for Exercise (1d).

3. We list those constraints that are binding for each LP.

   (a) Constraints 2 and 3

TABLE 2.6: Tableau after first iteration

z	x_1	x_2	s_1	s_2	s_3	RHS
1	0	-3	4	0	0	16
0	1	0	1	0	0	4
0	0	1	2	1	0	20
0	0	2	-1	0	1	10

TABLE 2.7: Final tableau

z	x_1	x_2	s_1	s_2	s_3	RHS
1	0	0	5/2	0	3/2	31
0	1	0	1	0	0	4
0	0	0	5/2	1	-1/2	15
0	0	1	-1/2	0	1/2	5

(b) Constraints 1 and 3

(c) Constraints 1 and 2

(d) Constraint 2

Solutions to Waypoints

Waypoint 2.1.1. We first state the LP in terms of maximization. If we set $\tilde{z} = -z$, then

$$\begin{aligned} \text{maximize } \tilde{z} &= 5x_1 - 2x_2 \\ \text{subject to} & \\ x_1 \quad\;\; &\leq 2 \\ 3x_1 + 2x_2 &\leq 6 \\ 8x_1 + 3x_2 &\leq 12 \\ x_1, x_2 &\geq 0, \end{aligned} \tag{2.1}$$

where $\tilde{z} = -z$.
The initial tableau corresponding to LP (2.1) is given by
Since we are now solving a maximization problem, we pivot on the highlighted entry in this tableau, which yields the following:
All entries in the top row corresponding to nonbasic variables are positive. Hence $x_1 = \frac{3}{2}$ and $x_2 = 0$ is a solution of the LP. However, the objective value corresponding to this optimal solution, for the original LP, is given by $z = -\tilde{z} = -\frac{15}{2}$.

TABLE 2.8: Initial tableau

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	-4	-1	-5	0	0	0	0
0	2	1	3	1	0	0	14
0	6	3	3	0	1	0	22
0	2	3	0	0	0	1	14

TABLE 2.9: Tableau after first iteration

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	-2/3	2/3	0	5/3	0	0	70/3
0	2/3	1/3	1	1/3	0	0	14/3
0	4	2	0	-1	1	0	8
0	2	3	0	0	0	1	14

TABLE 2.10: Final tableau

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	0	1	0	3/2	1/6	0	74/3
0	0	0	1	1/2	-1/6	0	10/3
0	1	1/2	0	-1/4	1/4	0	2
0	0	2	0	1/2	-1/2	1	10

TABLE 2.11: Initial tableau

z	x_1	x_2	s_1	s_2	RHS
1	-3	2	0	0	0
0	1	-2	1	0	2
0	1	1	0	1	4

TABLE 2.12: Final tableau

z	x_1	x_2	s_1	s_2	RHS
1	-5	0	0	-2	-8
0	3	0	1	2	10
0	1	1	0	1	4

TABLE 2.13: Initial tableau

z	x_1	x_2	s_1	s_2	s_3	RHS
1	1	-4	0	0	0	0
0	-1	-3	1	0	0	-8
0	-1	-1	0	1	0	-4
0	1	-1	0	0	1	2

TABLE 2.14: Initial tableau

z	x_1	x_2	s_1	s_2	s_3	RHS
1	-5	2	0	0	0	0
0	1	0	1	0	0	2
0	3	2	0	1	0	6
0	8	3	0	0	1	12

TABLE 2.15: Final tableau

z	x_1	x_2	s_1	s_2	s_3	RHS
1	0	31/8	0	0	5/8	15/2
0	0	-3/8	1	0	-1/8	1/2
0	0	7/8	0	1	-3/8	3/2
0	1	3/8	0	0	1/8	3/2

2.2 Alternative Optimal/Unbounded Solutions and Degeneracy

1. Exercise 1

   The initial tableau is shown in Table

   TABLE 2.16: Initial tableau for Exercise 1

z	x_1	x_2	s_1	s_2	RHS
1	-2	-6	0	0	0
0	1	3	1	0	6
0	0	1	0	1	1

   After performing two iterations of the simplex algorithm, we obtain the following:

   TABLE 2.17: Tableau after second iteration for Exercise 1

z	x_1	x_2	s_1	s_2	RHS
1	0	0	2	0	12
0	1	0	1	-3	3
0	0	1	0	1	1

   At this stage, the basic variables are $x_1 = 3$ and $x_2 = 1$, with a corresponding objective value $z = 12$. The nonbasic variable, s_2, has a coefficient of zero in the top row. Thus, letting s_2 become basic will not change the value of z. In fact, if we pivot on the highlighted entry in Table 2.17, x_2 replaces s_2 as a nonbasic variable, $x_1 = 6$, and the objective value is unchanged.

2. Exercise 2 For an LP minimization problem to be unbounded, there must

   exist, at some stage of the simplex algorithm, a basic feasible solution in which a nonbasic variable has a negative coefficient in the top row and nonpositive coefficients in the remaining rows.

3. Exercise 3

   (a) For the current basic solution to not be optimal, it must be the case that $a < 0$. However, if the LP is bounded, then the ratio test forces $b > 0$. Thus, we may pivot on row and column containing b, in which case, we obtain the following tableau:

TABLE 2.18: Tableau obtained after pivoting on row and column containing b

z	x_1	x_2	s_1	s_2	RHS
1	0	$-\frac{a}{b}$	0	$3-\frac{a}{b}$	$10-2\frac{a}{b}$
0	1	$\frac{1}{b}$	0	$\frac{1}{b}-1$	$3+\frac{2}{b}$
0	0	$\frac{1}{b}$	1	$\frac{1}{b}$	$\frac{2}{b}$

Thus, $x_1 = 3 + \frac{2}{b}$, $x_2 = 0$, and $z = 10 - 2\frac{a}{b}$.

(b) If the given basic solution was optimal, then $a \geq 0$. However, if the LP has alternative optimal solutions, then $a = 0$. In this case, pivoting on the same entry in the original tableau as we did in (b), we would obtain the result in Table 2.18, but with $a = 0$. Thus, $x_1 = 3 + \frac{2}{b}$, $x_2 = 0$, and $z = 10$.

(c) If the LP is unbounded, then $a < 0$ and $b \leq 0$.

4. Exercise 4

(a) The coefficient of the nonbasic variable, s_1, in the top row of the tableau is negative. Since the coefficient of s_1 in the remaining two rows is also negative, we see that the LP is unbounded.

(b) So long as s_2 remains non basic, the equations relating s_1 to each of x_1 and x_2 are given by $x_1 - 2s_1 = 4$ and $x_2 - s_1 = 5$. Thus, for each unit of increase in s_1, x_1 increases by 2 units; for each unit of increase in s_1, x_2 increases by 1 unit.

(c) Eliminating s_1 from the equations $x_1 - 2s_1 = 4$ and $x_2 - s_1 = 5$, we obtain $x_2 = \frac{1}{2}x_1 + 3$.

(d) When $s_1 = 0$, $x_1 = 4$ and $x_2 = 5$. Since $s_1 \geq 0$, $x_1 \geq 4$ and $x_2 \geq 5$. Thus,we sketch that portion of the line $x_2 = \frac{1}{2}x_1 + 3$ for which $x_1 \geq 4$ and $x_2 \geq 5$.

(e) Since $z = 1{,}000$, $z = x_1 + 3x_2$, and $x_2 = \frac{1}{2}x_1 + 3$, we have a system of equations whose solution is given by $x_1 = \frac{1982}{5}$ and $x_2 = \frac{1006}{5}$.

5. Exercise 5

(a) The LP has 2 decision variables and 4 constraints. Inspection of the graph shows that the constraints are given by

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 6 \\ \frac{1}{2}x_1 + x_2 &\leq 4 \\ -\frac{1}{2}x_1 + x_2 &\leq 2. \end{aligned}$$

By introducing slack variables, s_1, s_2, s_3, and s_4, we can convert the preceding list of inequalities to a system of equations in six variables, x_1, x_2, s_1, s_2, s_3, and s_4. Basic feasible solutions are then most easily determined by substituting each of the extreme points into the system of equations and solving for the slack variables. The results are as follows:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \\ 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 1 \\ 3/2 \\ 9/2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 8 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

Observe that the extreme point $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ yields slack variable values for which $s_1 = s_2 = s_3 = 0$. This means that when any two of these three variables are chosen to be nonbasic, the third variable, which is basic, must be zero. Hence, the LP is degenerate. From a graphical perspective, the LP is degenerate because the boundaries of three of the constraints intersect at the single extreme point, $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(b) If the origin is the initial basic feasible solution, at least two simplex algorithm iterations are required before a degenerate basic feasible solution results. This occurs when the objective coefficient of x_1 is larger than that of x_2. In this case, the intermediate iteration yields $\mathbf{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. If the objective coefficient of x_2 is positive and larger than that of x_1, then at most three iterations are required.

Solutions to Waypoints

Waypoint 2.2.1.

1. The current basic feasible solution consists of $x_1 = a$, $s_1 = 2$, and $x_2 = s_2 = 0$. The LP has alternative optimal solutions if a nonbasic variable has a coefficient of zero in the top row of the tableau. This occurs when $a = 1$ or $a = 2$.

2. The LP is unbounded if both $1 - a < 0$ and $a - 3 < 0$, i.e., if $1 < a \leq 3$. The LP is also unbounded if both $2 - a < 0$ and $a - 4 \leq 0$, i.e., if $2 < a \leq 4$.

2.3 Excess and Artificial Variables: The Big M Method

1. (a) We let e_1 and a_1 denote the excess and artificial variables, respectively, that correspond to the first constraint. Similarly, we let e_2 and a_2 correspond to the second constraint. Using $M = 100$, our objective becomes one of minimizing $z = -x_1 + 4x_2 + 100a_1 + 100a_2$. If s_3 is the slack variable corresponding to the third constraint our initial tableau is as follows:

TABLE 2.19: Initial tableau

z	x_1	x_2	e_1	a_1	e_2	a_2	s_3	RHS
1	1	-4	0	-100	0	-100	0	0
0	1	3	-1	1	0	0	0	8
0	1	1	0	0	-1	1	0	4
0	1	-1	0	0	0	0	1	2

To obtain an initial basic feasible solution, we pivot on each of the highlighted entries in this tableau. The result is the following:

TABLE 2.20: Tableau indicting initial basic feasible solution

z	x_1	x_2	e_1	a_1	e_2	a_2	s_3	RHS
1	201	396	-100	0	-100	0	0	1200
0	1	3	-1	1	0	0	0	8
0	1	1	0	0	-1	1	0	4
0	1	-1	0	0	0	0	1	2

Three iterations of the simplex algorithm are required before all coefficients in the top row of the tableau that correspond to nonbasic variables are nonpositive. The final tableau is given by

TABLE 2.21: Final tableau

z	x_1	x_2	e_1	a_1	e_2	a_2	s_3	RHS
1	0	0 -3/4	-397/4	0	-100	-7/4	5/2	
0	0	1	-1/4	1/4	0	0	-1/4	3/2
0	1	0	-1/4	1/4	0	0	3/4	7/2
0	0	0	-1/2	1/2	1	-1	1/2	1

The solution is $x_1 = \frac{7}{2}$ and $x_2 = \frac{3}{2}$, with $x_2 = \frac{5}{2}$.

(b) We let e_1 and a_1 denote the excess and artificial variables, respectively, that correspond to the first constraint. Similarly, we let e_3 and a_3 correspond to the third constraint. Using $M = 100$, our objective becomes one of minimizing $z = x_1 + x_2 + 100a_1 + 100a_3$. If s_2 is the slack variable corresponding to the second constraint our initial tableau is as follows:

TABLE 2.22: Initial tableau

z	x_1	x_2	e_1	a_1	s_2	e_3	a_3	RHS
1	-1	-1	0	-100	0	0	-100	0
0	2	3	-1	1	0	0	0	30
0	-1	2	0	0	1	0	0	6
0	1	3	0	0	0	-1	1	18

To obtain an initial basic feasible solution, we pivot on each of the highlighted entries in this tableau. The result is the following:

TABLE 2.23: Tableau indicting initial basic feasible solution

z	x_1	x_2	e_1	a_1	s_2	e_3	a_3	RHS
1	299	599	-100	0	0	-100	0	4800
0	2	3	-1	1	0	0	0	30
0	-1	2	0	0	1	0	0	6
0	1	3	0	0	0	-1	1	18

Three iterations of the simplex algorithm lead to a final tableau given by

TABLE 2.24: Final tableau

z	x_1	x_2	e_1	a_1	s_2	e_3	a_3	RHS
1	0	0	-3/7	-697/7	-1/7	0	-100	12
0	0	0	-5/7	5/7	3/7	1	-1	6
0	0	1	-1/7	1/7	2/7	0	0	6
0	1	0	-2/7	2/7	-3/7	0	0	6

The solution is $x_1 = 6$ and $x_2 = 6$, with $x_2 = 12$.

(c) We let a_1 denote the artificial variable corresponding to the first (equality) constraint, and we let e_2 and a_2 denote the excess and artificial variables, respectively, that correspond to the second constraint. Using $M = 100$, our objective becomes one of maximizing $z = x_1 + 5x_2 + 6x_3 - 100a_1 - 100a_2$. The initial tableau is given by

To obtain an initial basic feasible solution, we pivot on each of the highlighted entries in this tableau. The result is the following:

TABLE 2.25: Initial tableau

z	x_1	x_2	x_3	a_1	e_2	a_2	RHS
1	-1	-5	-6	100	0	100	0
0	1	4	2	1	0	0	50
0	1	-4	4	0	-1	1	40

TABLE 2.26: Tableau indicting initial basic feasible solution

z	x_1	x_2	x_3	a_1	e_2	a_2	RHS
1	-201	-5	-606	0	100	-9000	
0	1	4	2	1	0	0	50
0	1	-4	4	0	-1	1	40

Three iterations of the simplex algorithm are required before all coefficients in the top row of the tableau that correspond to nonbasic variables are nonnegative. The final tableau is given by

TABLE 2.27: Final tableau

z	x_1	x_2	x_3	a_1	e_2	a_2	RHS
1	2	7	0	103	0	100	150
0	1	12	0	2	1	-1	60
0	1/2	2	1	1/2	0	0	25

The solution is $x_1 = x_2 = 0$ and $x_3 = 25$, with $z = 150$.

Solutions to Waypoints

Waypoint 2.3.1. The LP is given by

$$\begin{aligned} \text{maximize } z &= \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} & \\ [A|I_2]\begin{bmatrix}\mathbf{x}\\ \mathbf{s}\end{bmatrix} &= \mathbf{b} \\ \mathbf{x}, \mathbf{s} &\geq \mathbf{0}, \end{aligned}$$

where $A = \begin{bmatrix} 1.64 & 2.67 \\ -2.11 & -2.3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -45.55 & -21.87 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 31.2 \\ -13.0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$.

Waypoint 2.3.2. Let s_1 denote the slack variable for the first constraint, and let e_2 and a_2 denote the excess and artificial variables, respectively, for the second constraint. Using $M = 100$, our objective becomes one of minimizing $z = 45.55x_1 + 21.87x_2 + 100a_2$. The initial tableau is as follows:

TABLE 2.28: Initial tableau

z	x_1	x_2	s_1	e_2	a_2	RHS
1	-45.55	-21.87	0	0	-100	0
0	1.64	2.67	1	0	0	31.2
0	2.11	2.3	0	-1	1	13.9

To obtain an initial basic feasible solution, we pivot on the highlighted entry in this tableau. The result is the following:

TABLE 2.29: Tableau indicting initial basic feasible solution

z	x_1	x_2	s_1	e_2	a_2	RHS
1	165	208	0	-100	0	1390
0	1.64	2.67	1	0	0	13.2
0	2.11	2.3	0	-1	1	13.9

Two iteration of the simplex algorithm yield the final tableau:

TABLE 2.30: Final tableau

z	x_1	x_2	s_1	e_2	a_2	RHS
1	-26	0	0	-9.5	-90.5	132.17
0	-.81	0	1	1.16	-1.16	15
0	.918	1	0	-.435	.435	6.04

Thus x_1 is nonbasic so that the total foraging time is minimized when $x_1 = 0$ grams of grass and $x_2 \approx 6.04$ grams of forb, in which case $z \approx 132.17$ minutes

2.4 Duality

1. At each stage of the simplex algorithm, the decision variables in the primal LP correspond to a basic feasible solutions. Values of the decision variables in the dual LP are recorded by the entries of the vector $\mathbf{y}$. Only at the final iteration of the algorithm, do the dual variables satisfy both $\mathbf{y}A \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$, i.e., only at the final iteration do the entries of $\mathbf{y}$ constitute a basic feasible solution of the dual.

2. The given LP can be restated as

$$\begin{aligned} \text{maximize } z &= 3x_1 + 4x_2 \\ \text{subject to}& \\ x_1 \quad &\leq 4 \\ x_1 + 3x_2 &\leq 15 \\ -x_1 + 2x_2 &\geq 5 \\ x_1 - x_2 &\geq 9 \\ x_1 + x_2 &\leq 6 \\ -x_1 - x_2 &\leq -6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

In matrix inequality form, this becomes

$$\begin{aligned} \text{maximize } z &= \mathbf{c} \cdot \mathbf{x} \\ \text{subject to}& \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}, \end{aligned}$$

where $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 15 \\ -5 \\ 9 \\ 6 \\ -6 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The corresponding dual is given by

$$\begin{aligned} \text{minimize } w &= \mathbf{y} \cdot \mathbf{b} \\ \text{subject to}& \\ \mathbf{y}A &\geq \mathbf{c} \\ \mathbf{y} &\geq \mathbf{0}, \end{aligned}$$

where $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix}$. In expanded form this is identical to

$$\begin{aligned} \text{minimize } w = 4y_1 + 15y_2 - 5y_3 + 9y_4 + 6(y_5 - y_6) \\ \text{subject to} \\ y_1 + y_2 + y_3 - y_4 + (y_5 - y_6) \geq 3 \\ 3y_2 - y_3 + y_4 + (y_5 - y_6) \\ y_1, y_2, y_3, y_4, y_5, y_6 \geq 0. \end{aligned}$$

Now define the new decision variable $\widetilde{y} = y_5 - y_6$, which is unrestricted in sign due to the fact y_5 and y_6 are nonnegative. Since y_5 and y_6, together, correspond to the equality constraint in the primal LP, we see that $\widetilde{y}$ is a dual variable unrestricted in sign that corresponds to an equality constraint in the primal.

3. The standard maximization LP can be written in matrix inequality form as

$$\begin{aligned} \text{maximize } z = \mathbf{c}^t\mathbf{x} \\ \text{subject to} \\ A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{x}$ belongs to $\mathbb{R}^n$, $\mathbf{c}$ is a <u>column</u> vector in $\mathbb{R}^n$, $\mathbf{b}$ belongs to $\mathbb{R}^m$, and A is an m-by-n matrix.

The dual LP is then expressed as

$$\begin{aligned} \text{minimize } w = \mathbf{y}^t\mathbf{b} \\ \text{subject to} \\ \mathbf{y}^t A \geq \mathbf{c}^t \\ \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

where we have assumed the dual variable vector, $\mathbf{y}$, is a column vector in $\mathbb{R}^m$. But, by transpose properties, this is equivalent to

$$\begin{aligned} \text{maximize } \widetilde{w} = -\mathbf{b}^t\mathbf{y} \\ \text{subject to} \\ -A^t\mathbf{y} \leq -\mathbf{c} \\ \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

The preceding LP is a maximization problem, whose dual is given by

$$\begin{aligned} \text{minimize } \widetilde{z} &= \mathbf{x}^t(-\mathbf{c}) \\ \text{subject to} & \\ \mathbf{x}^t(-A^t) &\geq -\mathbf{b}^t \\ \mathbf{x} &\geq \mathbf{0}. \end{aligned}$$

Using transpose properties again, we may rewrite this final LP as

$$\begin{aligned} \text{maximize } z &= \mathbf{c}^t\mathbf{x} \\ \text{subject to} & \\ A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}, \end{aligned}$$

which is the original LP.

4. (a) The original LP has four decision variables and three constraints. Thus, the dual LP has three decision variables and four constraints.
   (b) The current primal solution is not optimal, as the nonbasic variable, s_1, has a negative coefficient in the top row of the tableau. By the ratio test, we conclude that s_1 will replace x_1 as a basic variable and that $s_1 = 4$ in the updated basic feasible solution. Since the coefficient of s_1 in the top row is -2, the objective value in the primal will increase to 28. By weak duality, we can conclude that the objective value in the solution of the dual LP is no less than 28.

5. The dual of the given LP is

$$\begin{aligned} \text{minimize } w &= y_1 - 2y_2 \\ \text{subject to} & \\ y_1 - y_2 &\geq 2 \\ -y_1 + y_2 &\geq 1 \\ y_1, y_2 &\geq 0, \end{aligned}$$

which is infeasible.

6. (a) The current tableau can also be expressed in partitioned matrix form as

$$\begin{bmatrix} z & \mathbf{x} & \mathbf{y} & \mathbf{yb} \\ 1 & [\blacksquare, \blacksquare] & [1,0,0] & \blacksquare \\ \mathbf{0} & MA & M & M\mathbf{b} \end{bmatrix}$$

Comparing this matrix to that given in the problem, we see that

$M = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/2 & 1^0 & \\ -1/3 & 0 & 1 \end{bmatrix}$. Thus,

$$\begin{aligned} \mathbf{b} &= M^{-1}(M\mathbf{b}) \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 10 \end{bmatrix} \qquad = \begin{bmatrix} 18 \\ 12 \\ 16 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} A &= M^{-1}(MA) \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2/3 & 1 \\ 3 & 0 \\ 14/3 & 0 \end{bmatrix} \qquad = \begin{bmatrix} -2 & 3 \\ 2 & 3/2 \\ 4 & 1 \end{bmatrix}. \end{aligned}$$

Using A and b, we obtain the original LP:

$$\begin{aligned} \text{maximize } z &= 2x_1 + 3x_2 \\ &\text{subject to} \\ -2x_1 + 3x_2 &\leq 18 \\ 2x_1 + \frac{3}{2}x_2 &\leq 12 \\ 4x_1 + x_2 &\leq 16 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(b) The coefficients of x_1 and x_2 in the top row of the tableau are given by

$$\begin{aligned} -\mathbf{c} + \mathbf{y}A &= -[2,3] + [1,0,0]A \\ &= [-4,0] \end{aligned}$$

The unknown objective value is

$$\begin{aligned} z &= \mathbf{y}\mathbf{b} \\ &= [1,0,0] \begin{bmatrix} 18 \\ 12 \\ 16 \end{bmatrix} \\ &= 18. \end{aligned}$$

(c) The coefficient of x_1 in the top row of the tableau is -4, so an additional iteration of the simplex algorithm is required. Performing the ratio test, we see that x_1 replaces s_2 as a nonbasic variable. The updated decision variables become $x_1 = 1$ and $x_2 = \frac{20}{3}$, with a corresponding objective value of $z = 22$. The slack variable coefficients in the top row of the tableau are $s_1 = \frac{1}{3}$, $s_2 = \frac{4}{3}$, and $s_3 = 0$, which indicates that the current solution is optimal. Therefore the dual solution is given by $\mathbf{y} = \begin{bmatrix} 1/3 \\ 4/3 \\ 0 \end{bmatrix}$ with objective value $w = 22$.

7. Since the given LP has two decision variables and four constraints, the corresponding dual has four decision variables, y_1, y_2, y_3, and y_4 along with two constraints. In expanded form, it is given by

$$
\begin{aligned}
\text{minimize } z &= 4y_1 + 6y_2 + 33y_3 + 24y_4 \\
\text{subject to} & \\
-y1 + 2y_3 + 2y_4 &\geq 1 \\
y_1 + y_2 + 3y_3 + y_4 &\geq 5 \\
y_1, y_2, y_3, y_4 &\geq 0.
\end{aligned}
$$

To solve the dual LP using the simplex algorithm, we subtract excess variables, e_1 and e_2, from the first and second constraints, respectively, and add artificial variables, a_1 and a_2. The Big M method with $M = 1000$ yields an initial tableau given by

TABLE 2.31: Initial tableau

w	y_1	y_2	y_3	y_4	e_1	a_1	e_2	a_2	RHS
1	-4	-6	-33	-24	0	-1000	0	-1000	0
0	-1	0	2	2	-1	1	0	0	1
0	1	1	3	1	0	0	-1	1	5

.

The final tableau is

TABLE 2.32: Final tableau

w	y_1	y_2	y_3	y_4	e_1	a_1	e_2	a_2	RHS
1	-11/2	0	0	-3	-15/2	-1985/2	-6	-994	75/2
0	-1/2	0	1	1	-1/2	1/2	0	0	1/2
0	5/2	1	0	-2	3/2	-3/2	-1	1	7/2

.

The excess variable coefficients in the top row of this tableau are the additive inverses of the decision variable values in the solution of the primal. Hence, the optimal solution of the primal is given by $x_1 = \dfrac{15}{2}$ and $x_2 = 6$, with corresponding objective value $z = \dfrac{75}{2}$.

8. By the complementary slackness property, $\mathbf{x}_0$ is the optimal solution of the LP provided we can find $\mathbf{y}_0 = \begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_4 \end{bmatrix}$ in $\mathbb{R}^4$ such that

$$[\mathbf{y}_0 A - \mathbf{c}]_i \, [\mathbf{x}_0]_i = 0 \quad 1 \le i \le 6$$

and

$$[\mathbf{y}_0]_j \, [\mathbf{b} - A\mathbf{x}_0]_j = 0 \quad 1 \le j \le 4.$$

We have

$$\mathbf{b} - A\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 61/7 \\ 0 \end{bmatrix},$$

so that in a dual solution it must be the case that $y_3 = 0$. Furthermore, when $y_3 = 0$,

$$\mathbf{y}_0 A - \mathbf{c} = \begin{bmatrix} y_1 + 7y_2 + 7y_4 - 5 \\ 2y_1 - 5y_2 + 8y_4 - 1 \\ 4y_1 + 2y_2 - 3y_4 - 1 \\ 3y_1 + y_2 + 5y_4 - 4 \\ 4y_1 + 2y_2 + 2y_4 - 1 \\ 6y_1 + 3y_2 + 4y_4 - 2 \end{bmatrix}.$$

By complementary slackness again, the first, third and fourth components of this vector must equal zero, whence we have a system of 3 linear equations, whose solution is given by $y_1 = \frac{103}{204}$, $y_2 = \frac{37}{204}$, and $y_3 = \frac{47}{102}$. If we define $\mathbf{y}_0 = \begin{bmatrix} 103/204 \\ 37/204 \\ 0 \\ 47/102 \end{bmatrix}$, then

$$\begin{aligned} w &= \mathbf{y}_0 \mathbf{b} \\ &= \frac{53}{17} \\ &= \mathbf{c}\mathbf{x}_0 \end{aligned}$$

so that $\mathbf{x}_0$ is the optimal solution of the LP by Strong Duality.

9. Player 1's optimal mixed strategy is given by the solution of the LP

$$\begin{aligned} \text{maximize } z \text{ subject to} \\ A\mathbf{x} \geq z\mathbf{e} \\ \mathbf{e}^t \cdot \mathbf{x} = 1 \\ \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

and Player 2's by the solution of the corresponding dual LP,

$$\begin{aligned} \text{minimize } w \text{ subject to} \\ \mathbf{y}A \leq w\mathbf{e}^t \\ \mathbf{y} \cdot \mathbf{e} = 1 \\ \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Here $\mathbf{e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The solution of the first LP is given by $\mathbf{x}_0 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and the solution of the second by $\mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. The corresponding objective value for each LP is $z = w = .5$, thereby indicating that the game is biased in Player 1's favor.

10. (a) Since A is an m-by-n matrix, $\mathbf{b}$ belongs to $\mathbb{R}^m$, and $\mathbf{c}$ is a row vector in $\mathbb{R}^n$, the matrix M has $m+n-1$ rows and $m+n-1$ columns. Furthermore, the transpose of a partitioned matrix, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, is given by $\begin{bmatrix} A^t & C^t \\ B^t & D^t \end{bmatrix}$, which can be used to show $M^t = -M$.

    (b) Define $vecw_0 = \begin{bmatrix} \mathbf{y}_0^t \\ \mathbf{x}_0 \\ 1 \end{bmatrix}$. Then $\mathbf{w}_0$ belongs to $\mathbb{R}^{m+n+1}$ and has nonnegative components since $\mathbf{x}_0$ and $\mathbf{y}_0$ are primal feasible and dual feasible vectors belonging to $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Furthermore, $[\mathbf{w}_0]_{m+n+1} = 1 > 0$. We have

$$\begin{aligned} M\mathbf{w}_0 &= \begin{bmatrix} 0_{m\times m} & -A & \mathbf{b} \\ A^t & 0_{m\times n} & -\mathbf{c}^t \\ -\mathbf{b}^t & \mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_0^t \\ \mathbf{x}_0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -A\mathbf{x}_0 + \mathbf{b} \\ A^t\mathbf{y}_0^t - \mathbf{c}_t \\ -\mathbf{b}^t\mathbf{y}_0^t + \mathbf{c}\mathbf{x}_0 \end{bmatrix} \\ &\geq \mathbf{0} \end{aligned}$$

since the primal feasibility of $\mathbf{x}_0$ dictates $A\mathbf{x}_0 \leq \mathbf{b}$, the dual feasibility of $\mathbf{y}_0$ and transpose properties imply $A^t\mathbf{y}_0^t \geq \mathbf{c}_t$, and strong duality forces $\mathbf{b}^t\mathbf{y}_0^t = \mathbf{c}\mathbf{x}_0$.

(c) Letting κ, $\mathbf{x}_0$, and $\mathbf{y}_0$ be defined as in the hint, we have

$$\begin{aligned}
\mathbf{0} &\leq M\mathbf{w}_0 \\
&= \begin{bmatrix} 0_{m\times m} & -A & \mathbf{b} \\ A^t & 0_{m\times n} & -\mathbf{c}^t \\ -\mathbf{b}^t & \mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} \kappa\mathbf{y}_0^t \\ \kappa\mathbf{x}_0 \\ \kappa \end{bmatrix} \\
&= \kappa \begin{bmatrix} 0_{m\times m} & -A & \mathbf{b} \\ A^t & 0_{m\times n} & -\mathbf{c}^t \\ -\mathbf{b}^t & \mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_0^t \\ \mathbf{x}_0 \end{bmatrix} \\
&= \kappa \begin{bmatrix} -A\mathbf{x}_0 + \mathbf{b} \\ A^t\mathbf{y}_0^t - \mathbf{c}_t \\ -\mathbf{b}^t\mathbf{y}_0^t + \mathbf{c}\mathbf{x}_0 \end{bmatrix}
\end{aligned}$$

Since $\kappa > 0$, we conclude that $A\mathbf{x}_0 \leq \mathbf{b}$, $A^t\mathbf{y}_0^t \geq \mathbf{c}^t$, and $\mathbf{c}\mathbf{x}_0 \geq \mathbf{b}^t\mathbf{y}_0^t$. Transpose properties permit us to rewrite the second of these matrix inequalities as $\mathbf{y}_0 A \geq \mathbf{c}$. Thus $\mathbf{x}_0$ and $\mathbf{y}_0$ are primal feasible and dual feasible, respectively. Moreover, the third inequality is identical to $\mathbf{c}\mathbf{x}_0 \geq \mathbf{y}_0\mathbf{b}$ if we combine the facts $\mathbf{b}^t\mathbf{y}_0^t = \mathbf{y}_0\mathbf{b}$, each of these two quantities is a scalar, and the transpose of a scalar is itself. But the Weak Duality Theorem also dictates $\mathbf{c}\mathbf{x}_0 \leq \mathbf{y}_0\mathbf{b}$, which implies $\mathbf{c}\mathbf{x}_0 = \mathbf{y}_0\mathbf{b}$. From this result, we conclude $\mathbf{x}_0$ and $\mathbf{y}_0$ constitute the optimal solutions of the (4.1) and (4.3), respectively.

Solutions to Waypoints

Waypoint 2.4.1. In matrix inequality form, the given LP can be written as

$$\begin{aligned}
&\text{maximize } z = \mathbf{c}\cdot\mathbf{x} \\
&\text{subject to} \\
&\qquad A\mathbf{x} \leq \mathbf{b} \\
&\qquad \mathbf{x} \geq \mathbf{0},
\end{aligned}$$

where $vecx = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{c} = [3, 5, 2]$, $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 1 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} = 2 \\ 2 \\ 4 \\ 3 \end{bmatrix}$. The dual LP is given by

$$\begin{aligned}\text{minimize } w &= \mathbf{y}\cdot\mathbf{b}\\ \text{subject to}&\\ \mathbf{y}A &\geq \mathbf{c}\\ \mathbf{y} &\geq \mathbf{0},\end{aligned}$$

where $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}$. In expanded form, this is equivalent to

$$\begin{aligned}\text{minimize } w = 2y_1 + 2y_2 + 4y_3 + 3y_4&\\ \text{subject to}&\\ 2y_1 + y_2 + 3y_3 - y_4 &\geq 3\\ y_1 + 2y_2 + y_3 + y_4 &\geq 5\\ y_1 + y_2 + 5y_3 + y_4 &\geq 2\\ y_1, y_2, y_3, y_4 &\geq 0.\end{aligned}$$

The original LP has its solution given by $\mathbf{x}_0 = \begin{bmatrix} 2/3 \\ 2/3 \\ 0 \end{bmatrix}$ with $z_0 = \frac{16}{3}$. The solution of the dual LP is $\mathbf{y}_0 = \begin{bmatrix} 1/3 & 7/3 & 0 & 0 \end{bmatrix}$ with $w_0 = \frac{16}{3}$.

Waypoint 2.4.2. 1. If Steve always chooses row one and Ed chooses columns one, two, and three with respective probabilities, x_1, x_2, and x_3, then Ed's expected winnings are given by

$$\begin{aligned} f_1(x_1, x_2, x_3) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= x_1 - x_2 + 2x_3, \end{aligned}$$

which coincides with the first entry of the matrix-vector product, $A\mathbf{x}$. By similar reasoning,

$$\begin{aligned} f_2(x_1, x_2, x_3) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= 2x_1 + 4x_2 - x_3 \end{aligned}$$

and

$$\begin{aligned} f_3(x_1, x_2, x_3) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= -2x_1 + 2x_3, \end{aligned}$$

which equal the second and third entries of $A\mathbf{x}$, respectively.

2. If Ed always chooses column one and Steve chooses rows one, two, and three with respective probabilities, y_1, y_2, and y_3, then Steve's expected winnings are given by

$$g_1(y_1, y_2, y_3) = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad = y_1 + 2y_2 - 2y_3,$$

which coincides with the first entry of the product, $\mathbf{y}A$. By similar reasoning,

$$\begin{aligned} g_2(y_1, y_2, y_3) &= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= -y_1 + 4y_2 \end{aligned}$$

and

$$\begin{aligned} g_3(y_1, y_2, y_3) &= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= 2y_1 - y_2 + 2y_3, \end{aligned}$$

which equal the second and third entries of $\mathbf{y}A$, respectively.

Waypoint 2.4.3. To assist in the formulation of the dual, we view the primal objective, z, as the difference of two nonnegative decision variables, z_1 and z_2. Thus the primal LP can be expressed in matrix inequality form as

$$\begin{aligned} &\text{maximize } \begin{bmatrix} 1 & -1 & \mathbf{0}_{1\times 3} \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ \mathbf{x} \end{bmatrix} \\ &\text{subject to} \\ &\begin{bmatrix} \mathbf{e} & -\mathbf{e} & -A \\ 0 & 0 & \mathbf{e}^t \\ 0 & 0 & -\mathbf{e}^t \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ \mathbf{x} \end{bmatrix} \leq \begin{bmatrix} \mathbf{0}_{3\times 1} \\ 1 \\ -1 \end{bmatrix} \\ &\mathbf{x} \geq \mathbf{0} \text{ and } z_1, z_2 \geq 0. \end{aligned}$$

If the dual LP has its vector of decision variables denoted by $\begin{bmatrix} \mathbf{y} & w_1 & w_2 \end{bmatrix}$, where $\mathbf{y}$ is a row vector in $\mathbb{R}^3$ with nonnegative components, and both w_1 and w_2 are nonnegative as well, then the dual is given by

$$\text{minimize } \begin{bmatrix} \mathbf{y} & w_1 & w_2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{0}_{3\times1} \\ 1 \\ -1 \end{bmatrix}$$

subject to

$$\begin{bmatrix} \mathbf{y} & w_1 & w_2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{e} & -\mathbf{e} & -A \\ 0 & 0 & \mathbf{e}^t \\ 0 & 0 * -\mathbf{e}^t & \end{bmatrix} \geq \begin{bmatrix} 1 & -1 & \mathbf{0}_{1\times3} \end{bmatrix}$$

$$\mathbf{y} \geq \mathbf{0}_{1\times3} \text{ and } w_1, w_2 \geq 0.$$

If we write $w = w_1 - w_2$, then this formulation to the dual simplifies to be

$$\text{minimize } w \text{ subject to}$$
$$\mathbf{y}A \leq w\mathbf{e}^t$$
$$\mathbf{y} \cdot \mathbf{e} = 1$$
$$\mathbf{y} \geq \mathbf{0}.$$

2.5 Sufficient Conditions for Local and Global Optimal Solutions

1. If $f(x_1, x_2) = e^{-\left(x_1^2+x_2^2\right)}$, then direct computations lead to the following results:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} -2x_1 f(x1, x2) \\ -2x_2 f(x_1, x_2) \end{bmatrix}$$

and

$$H_f(\mathbf{x}) = \begin{bmatrix} (-2+4x_1^2)f(x1, x2) & 4x_1x_2f(x_1, x_2) \\ 4x_1x_2f(x_1, x_2) & (-2+4x_2^2)f(x1, x2) \end{bmatrix}$$

so that

$$\nabla f(\mathbf{x}_0) \approx \begin{bmatrix} -.2707 \\ -.2707 \end{bmatrix}$$

and

$$H_f(\mathbf{x}_0) \approx \begin{bmatrix} .2707 & .5413 \\ .5413 & .2702 \end{bmatrix}.$$

Thus,

$$\begin{aligned} Q(\mathbf{x}) &= f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^t(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^t H_f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ &\approx 2.3007 - 1.895x_1 - 1.895x_2 + .2707x_1^2 + .2707x_2^2 + 1.0827x_1x_2 \end{aligned}$$

The quadratic approximation together with f are graphed in Figure 2.1. The solution is given by $\mathbf{x} = \begin{bmatrix} 4.5 \\ 0 \end{bmatrix}$, with $z = 27$.

2. Calculations establish that Hessian of f is given by $H_f(\mathbf{x}) = \phi(\mathbf{x})A$, where $\phi(\mathbf{x}) = \dfrac{1}{(74 + .5x_1 + x_2 + x_3)^{1/10}}$ and where

$$A = \begin{bmatrix} .045 & .09 & .09 \\ .09 & .19 & .19 \\ .09 & .19 & .19 \end{bmatrix}.$$

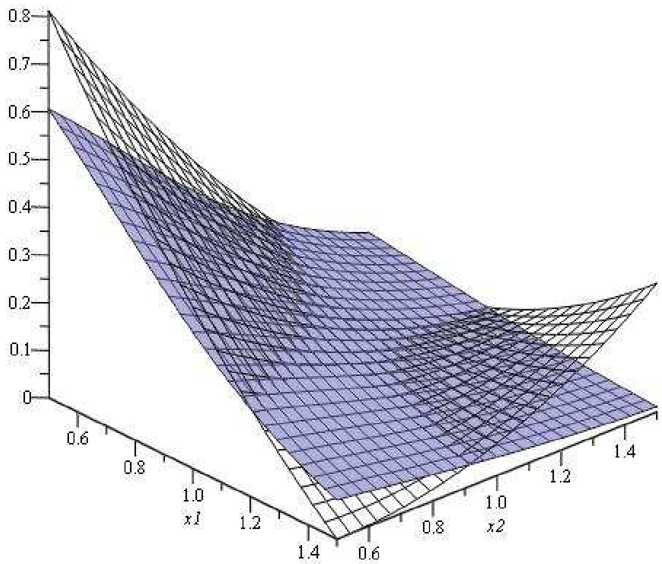

FIGURE 2.1: Plot of f with quadratic approximation

Observe that $\phi(\mathbf{x}) > 0$ for $\mathbf{x}$ in S and that A is positive semidefinite. (Its eigenvalues are .002, .423, and zero.)

Now suppose that $\mathbf{x}_0$ belongs to S and that λ_0 is an eigenvalue of $H_f(\mathbf{x}_0)$ with corresponding eigenvector $\mathbf{v}$. Then

$$\begin{aligned} \lambda_0 \mathbf{v} &= H_f(\mathbf{x}_0)\mathbf{v} \\ &= \phi(\mathbf{x}_0) A \mathbf{v}, \end{aligned}$$

implying $\dfrac{\lambda_0}{\phi(\mathbf{x}_0)}$ is an eigenvalue of A. Since $\phi(\mathbf{x}_0) > 0$ and each eigenvalue of A is nonnegative, then λ_0 must be nonnegative as well. Hence, $H_f(\mathbf{x}_0)$ is positive semidefinite, and f is convex.

3. (a) We have $\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 + x_2 - 1 \\ x_1 + 4x_2 - 4 \end{bmatrix}$, which yields the critical point, $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since $H_f(\mathbf{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ has positive eigenvalues, $\lambda = 3 \pm \sqrt{2}$, we see that f is strictly convex on $\mathbb{R}^2$ so that $\mathbf{x}_0$ is a global minimum.

(b) We have $\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 - 10x_2 + 4 \\ -10x_1 + 14x_2 - 8 \end{bmatrix}$, which yields the critical point, $\mathbf{x}_0 = \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix}$. Since $H_f(\mathbf{x}) = \begin{bmatrix} 2 & -10 & -10 & 14 \end{bmatrix}$ has mixed-sign eigenvalues, $\lambda = 8 \pm 2\sqrt{34}$, $\mathbf{x}_0$ is a saddle point.

(c) We have $\nabla f(\mathbf{x}) = \begin{bmatrix} -4x_1 + 6x_2 - 6 \\ 6x_1 - 10x_2 + 8 \end{bmatrix}$, which yields the critical point, $\mathbf{x}_0 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$. Since $H_f(\mathbf{x}) = \begin{bmatrix} -4 & 6 \\ 6 & -10 \end{bmatrix}$ has negative eigenvalues, $\lambda = -7 \pm 3\sqrt{5}$, we see that f is strictly concave on $\mathbb{R}^2$ so that $\mathbf{x}_0$ is a global maximum.

(d) We have $\nabla f(\mathbf{x}) = \begin{bmatrix} 4x_1^3 - 24x_1^2 + 48x_1 - 32 \\ 8x_2 - 4 \end{bmatrix}$, which yields a single critical point at $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$. The Hessian simplifies to $H_f(\mathbf{x}) = \begin{bmatrix} 12(x_1 - 2)^2 & 0 \\ 0 & 8 \end{bmatrix}$ and has eigenvalues of $12(x_1 - 2)^2$ and 8. Thus, f is concave on $\mathbb{R}^2$ and $\mathbf{x}_0$ is a global maximum.

(e) We have $\nabla f(\mathbf{x}) = \begin{bmatrix} \cos(x_1)\cos(x_2) \\ -\sin(x_1)\sin(x_2) \end{bmatrix}$, which yields four critical points: $\mathbf{x}_0 = \pm \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$ and $\mathbf{x}_0 = \pm \begin{bmatrix} 0 \\ \pi/2 \end{bmatrix}$. The Hessian is

$$H_f(\mathbf{x}) = \begin{bmatrix} -\sin(x_1)\cos(x_2) & -\cos(x_1)\sin(x_2) \\ -\cos(x_1)\sin(x_2) & -\sin(x_1)\cos(x_2) \end{bmatrix},$$

which yields a repeated eigenvalue, $\lambda = -1$, when $\mathbf{x}_0 = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$, a repeated eigenvalue $\lambda = 1$, when $\mathbf{x}_0 = \begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}$, and eigenvalues $\lambda = \pm 1$, when $\mathbf{x}_0 = \begin{bmatrix} 0 \\ \pm\pi/2 \end{bmatrix}$. Thus, $\mathbf{x}_0 = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$ is a local maximum, $\mathbf{x}_0 = \begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}$ is a local minimum, and $\mathbf{x}_0 = \begin{bmatrix} 0 \\ \pm\pi/2 \end{bmatrix}$ are saddle points.

(f) We have $\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1/(x_1^2 + 1) \\ x_2 \end{bmatrix}$, which yields a single critical point at the origin. Since f is differentiable and nonnegative on $\mathbb{R}^2$, we conclude immediately that the origin is a global minimum. In addition,

$$H_f(\mathbf{x}) = \begin{bmatrix} \frac{2(1-x_1)^2}{(x_1^2+1)^2} & 0 \\ 0 & \\ 1 & \end{bmatrix}.$$

The eigenvalues of this matrix are positive when $-1 < x_1 < 1$. Thus f is is strictly convex its given domain.

(g) If $f(x_1, x_2) = e^{-\left(\frac{x_1^2+x_2^2}{2}\right)}$, then $\nabla f(\mathbf{x}) = f(\mathbf{x})\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that f has a single critical point at the origin. The Hessian simplifies to $H_f(\mathbf{x}) = f(\mathbf{x})\begin{bmatrix} x_1^2 - 1 & x_1x_2 \\ x_1x_2 & \\ x_2^2 - 1 & \end{bmatrix}$, which has eigenvalues in terms of $\mathbf{x}$ given by $\lambda = \begin{bmatrix} -f(\mathbf{x}) \\ (x_1^2 + x_2^2 - 1)f(\mathbf{x}) \end{bmatrix}$. Since $f(\mathbf{x}) > 0$ and $x_1^2 + x_2^2 < 1$, we have that f is strictly concave on its given domain and the origin is a global minimum.

4. We have $\nabla f(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$, which is the zero vector precisely when $\mathbf{x}_0 = A^{-1}\mathbf{b}$. The Hessian of f is simply A, so f is strictly convex (resp. strictly concave) on $\mathbb{R}^n$ when A is positive definite (resp. negative definite). Thus, $\mathbf{x}_0$ is the unique global minimum (resp. unique global maximum) of f.

When $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $c = 6$, $\mathbf{x}_0 = \begin{bmatrix} -7/5 \\ 9/5 \end{bmatrix}$. The eigenvalues of A are $\lambda = \dfrac{5 \pm \sqrt{5}}{2}$, so $\mathbf{x}_0$ is the global minimum.

5. Write $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$. Since $\mathbf{n} \neq \mathbf{0}$, we lose no generality by assuming $n_3 \neq 0$. Solving $\mathbf{n}^t\mathbf{x} = d$ for x_3 yields x_3 as a function of x_1 and x_2:

$$f : \mathbb{R}^2 \to \mathbb{R} \text{ where } f(x_1, x_2) = x_1^2 + x_2^2 + \frac{(d - n_1x_1 - n_2x_2)^2}{n_3^2}.$$

The critical point of f is most easily computed using Maple and simplifies to $\mathbf{x}_0 = \left[n_1 d/\|\mathbf{n}\|^2, n_1 d/\|\mathbf{n}\|^2\right]$ The eigenvalues of the Hessian are $\lambda_1 = 2$ and $\lambda_2 = \frac{2\|\mathbf{n}\|^2}{n_3^2}$ so that f is strictly convex and $\mathbf{x}_0$ is the global minimum. The corresponding shortest distance is $\frac{|d|}{\|\mathbf{n}\|}$

For the plane $x_1+2x_2+x_3 = 1$, $\mathbf{n} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $d = 1$. In this case, $\mathbf{x}_0 = \begin{bmatrix}1/14\\1/7\\3/14\end{bmatrix}$ with a corresponding distance of $\dfrac{1}{\sqrt{14}}$.

6. The gradient of f is $\nabla f(\mathbf{x}) = \begin{bmatrix}4x_1^3\\2x_2\end{bmatrix}$, which vanishes at the origin. Since f is nonnegative and $f(\mathbf{0}) = 0$, we see that $\mathbf{x}_0 = \mathbf{0}$ is the global minimum. The Hessian of f at the origin is $H_f(\mathbf{0}) = \begin{bmatrix}0 & 0 & 0 & 2\end{bmatrix}$, which is positive semidefinite.

   The function $f(x_1, x_2) = -x_1^4 + x_2^2$ has a saddle point at the origin because $f(x_1, 0) = -x_1^4$ has a minimum at $x_1 = 0$ and $f(0, x_2) = x_2^2$ has a maximum at $x_2 = 0$. The Hessian of f at the origin is again $H_f(\mathbf{0}) = \begin{bmatrix}0 & 0 & 0 & 2\end{bmatrix}$.

7. The function $f(\mathbf{x}) = x_1^4 + x_2^4$ has a single critical point at $\mathbf{x}_0 = \mathbf{0}$, which is a global minimum since f is nonnegative and $f(\mathbf{0}) = 0$. The Hessian evaluated at $\mathbf{x}_0$ is the zero matrix, whose only eigenvalue is zero. Similarly, $f(\mathbf{x}) = -x_1^4 - x_2^4$ has a global maximum at $\mathbf{x}_0 = \mathbf{0}$, where its Hessian is also the zero matrix.

8. (a) Player 1's optimal mixed strategy is the solution of

$$\begin{aligned} &\text{maximize } = z \\ &\text{subject to} \\ &2x_1 - 3x_2 \geq z \\ &-x_1 + 4x_2 \geq z \\ &x_1 + x_2 = 1 \\ &x_1, x_2 \geq 0, \end{aligned}$$

   which is given by $\mathbf{x}_0 = \begin{bmatrix}7/10\\3/10\end{bmatrix}$, with corresponding objective value $z_0 = \dfrac{1}{2}$.

   Player 2's optimal mixed strategy is the solution of the corresponding dual LP,

$$\begin{aligned} &\text{minimize } = w \\ &\text{subject to} \\ &2y_1 - y_2 \leq w \\ &-3y_1 + 4y_2 \leq w \\ &y_1 + y_1 = 1 y_1, y_2 \qquad \geq 0, \end{aligned}$$

which is $\mathbf{y}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, with corresponding objective value $w_0 = \frac{1}{2}$.

(b) We have

$$\begin{aligned} f(x_1, y_1) &= \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix}^t A \begin{bmatrix} x_1 \\ 1 - x_1 \end{bmatrix} \\ &= 10x_1y_1 - 5x_1 - 7y_1 + 4, \end{aligned}$$

whose gradient, $\nabla f(x_1, y_1)$ vanishes at $x_1 = \frac{7}{10}$, $y_1 = \frac{1}{2}$. The Hessian of f is a constant matrix having eigenvalues $\lambda = \pm 10$ so that this critical point is a saddle point. The game value is $f\left(\frac{7}{10}, \frac{1}{2}\right) = \frac{1}{2}$.

(c) The only solution of $f(x_1, y_1) = \frac{1}{2}$ is simply $\left(\frac{7}{10}, \frac{1}{2}\right)$.

(d) A game value 20% higher than $\frac{1}{2}$ is $\frac{3}{5}$. The equation $f(x_1, y_1) = \frac{3}{5}$ simplifies to

$$10x_1y_1 - 5x_1 - 7y_1 + \frac{17}{5} = 0.$$

The graph of this relation is shown in Figure **??**.

9. Calculations, which are most easily performed with Maple, establish that $f(x_1, y_1) = \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix}^t A \begin{bmatrix} x_1 \\ 1 - x_1 \end{bmatrix}$ has a saddle point at $x_1 = \frac{d - b}{a + d - b - c}$, $y_1 = \frac{d - c}{a + d - b - c}$. Since

$$f\left(\frac{d - b}{a + d - b - c}, \frac{d - c}{a + d - b - c}\right) = \frac{ad - bc}{a + d - b - c},$$

the game is fair provided $\det A \neq 0$, or, in other words, if A is noninvertible.

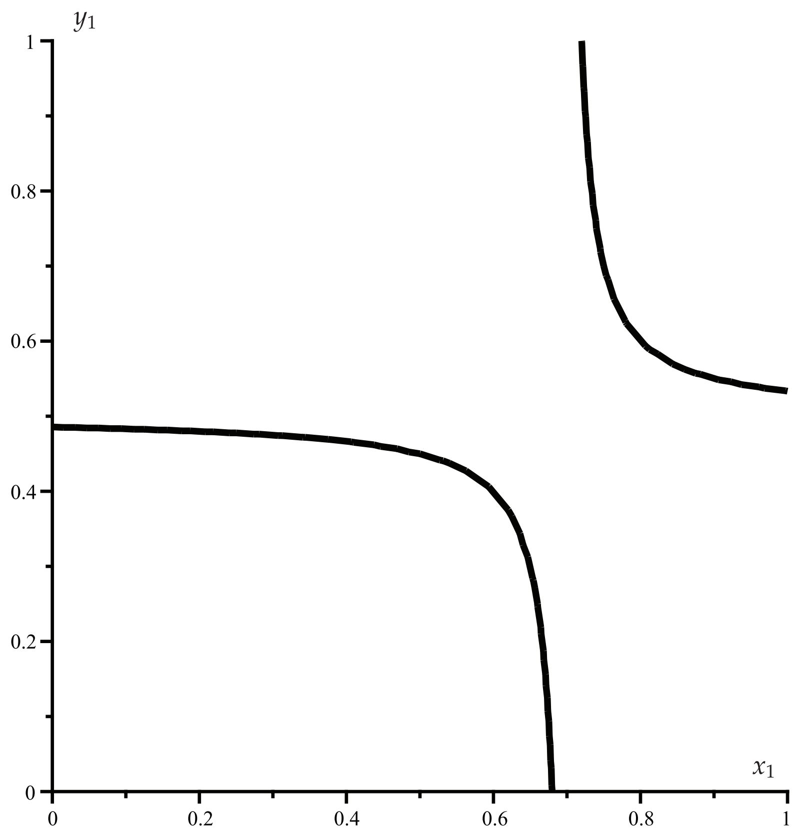

FIGURE 2.2: The relation $10x_1y_1 - 5x_1 - 7y_1 + \frac{17}{5} = 0$.

2.6 Quadratic Programming

1. (a) The Hessian of f is $\begin{bmatrix} 10 & 0 & 4 \\ 0 & 16 & 0 \\ 4 & 0 & 2 \end{bmatrix}$. If $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, then the problem is given as

$$\begin{aligned} &\text{minimize } f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t Q\mathbf{x} + \mathbf{p}^t\mathbf{x} \\ &\text{subject to} \\ &\qquad A\mathbf{x} = \mathbf{b}. \end{aligned}$$

(b) The eigenvalues of Q are $\lambda = 1$ (repeated) and $\lambda = 2$, so Q is positive definite and the problem is convex. Thus,

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\mu}_0 \\ \mathbf{x}_0 \end{bmatrix} &= \begin{bmatrix} 0_{m\times m} & A \\ A^t & Q \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b} \\ -\mathbf{p} \end{bmatrix} \\ &= \begin{bmatrix} -44/9 \\ 65/9 \\ -1/3 \\ -4/3 \\ 0 \end{bmatrix}. \end{aligned}$$

The problem then has its solution given by $\mathbf{x}_0 = \begin{bmatrix} 1/3 \\ -4/3 \\ 0 \end{bmatrix}$. The corresponding objective value is $f(1/3, -4/3) = \frac{106}{9}$.

2. (a) The matrix Q is indefinite. However, the partitioned matrix, $\begin{bmatrix} 0_{m\times m} & A \\ A^t & Q \end{bmatrix}$ is still invertible so that

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\mu}_0 \\ \mathbf{x}_0 \end{bmatrix} &= \begin{bmatrix} 0_{m\times m} & A \\ A^t & Q \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b} \\ -\mathbf{p} \end{bmatrix} \\ &= \begin{bmatrix} -259/311 \\ 178/311 \\ -191/311 \\ -11/311 \\ -142/311 \end{bmatrix}. \end{aligned}$$

Thus, the problem has a unique KKT point, $\mathbf{x}_0 = \begin{bmatrix} -191/311 \\ -11/311 \\ -142/311 \end{bmatrix}$, with corresponding multiplier vector, $\boldsymbol{\mu}_0 = \begin{bmatrix} -259/311 \\ 178/311 \end{bmatrix}$. The bordered Hessian becomes

$$B = \begin{bmatrix} 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -4 & 0 & 1 \\ -1 & -4 & 1 & -2 & 0 \\ 2 & 0 & -2 & 3 & 1 \\ -1 & 1 & 0 & 1 & 3 \end{bmatrix}.$$

In this case there are $n = 3$ decision variables and $m = 2$ equality constraints. Since the determinants of both B and its leading principal minor of order 4 are positive, we see that $\mathbf{x}_0$ is the solution.

(b) The problem has a unique KKT point given by $\mathbf{x}_0 = \begin{bmatrix} 6 \\ 13/4 \\ 1/4 \end{bmatrix}$ with corresponding multiplier vector, $\boldsymbol{\lambda}_0 = \begin{bmatrix} 5/2 \\ 3 \\ 0 \end{bmatrix}$. At $\mathbf{x}_0$ only the first two constraints of $C\mathbf{x} \leq \mathbf{d}$ are binding. Thus, we set $\tilde{C} = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & 3 \end{bmatrix}$ and $\tilde{\mathbf{d}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, in which case

$$B = \begin{bmatrix} 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & -1 & -3 \\ -1 & 1 & 1 & -2 & 0 \\ 2 & -1 & -2 & 3 & 1 \\ 2 & -3 & 0 & 1 & 3 \end{bmatrix}.$$

There are $n = 3$ decision variables, $m = 0$ equality constraints, and $k = 2$ binding inequality constraints. Since the determinant of B and its leading principal minor of order 4 are positive, we see that $\mathbf{x}_0$ is the solution.

(c) The are two KKT points. The first is given by $\mathbf{x}_0 = \begin{bmatrix} -17/40 \\ -1/10 \\ 7/8 \\ 0 \end{bmatrix}$ with corresponding multiplier vectors, $\boldsymbol{\lambda}_0 \approx \begin{bmatrix} 0 \\ 1.5788 \end{bmatrix}$ and $\boldsymbol{\mu}_0 \approx \begin{bmatrix} .4405 \\ -.2806 \\ 1.295 \end{bmatrix}$.

The second constraint is binding at $\mathbf{x}_0$, so we set $\tilde{C} = \begin{bmatrix} -1 & 3 & 1 & 0 \end{bmatrix}$ and $\tilde{\mathbf{d}} = \begin{bmatrix} 1 \end{bmatrix}$, in which case,

$$B = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 1} & A \\ 0_{1\times 3} & 0 & \tilde{C} \\ A^t & \tilde{C}^t & Q \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 01 & 2 & 3 & -1 & \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -5 & 4 & 2 \\ 0 & 0 & 0 & 0 & 3 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 & -1 & 3 & 1 & 0 \\ 1 & 0 & 3 & -1 & 2 & 4 & 0 & 0 \\ 2 & -5 & 1 & 3 & 4 & 7 & -2 & -1 \\ 3 & 4 & 5 & 1 & 0 & -2 & -3 & -1 \\ -1 & 2 & 6 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}$$

There are $n = 4$ decision variables, $m = 3$ equality constraints, and $k = 1$ binding inequality constraints. Since $n - m - k = 0$, we only need to check that the determinant of B itself has the same sign as $(-1)^{m+k} = 1$, which it does. Hence, $\mathbf{x}_0 = \begin{bmatrix} -17/40 \\ -1/10 \\ 7/8 \\ 0 \end{bmatrix}$ is a solution.

The second KKT point is given by $\mathbf{x}_0 \approx \begin{bmatrix} 7.593 \\ -2.406 \\ -.9591 \\ -2.096 \end{bmatrix}$ with corresponding multiplier vectors, $\boldsymbol{\lambda}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{\mu}_0 \approx \begin{bmatrix} -2.550 \\ 1.555 \\ -1.671 \end{bmatrix}$. In this case neither inequality constraint is binding. Performing an analysis similar to that done for the first KKT point leads to a situation in which the bordered Hessian test is inconclusive. In fact, $f(\mathbf{x}_0) \approx 12.1$, whereas $f\left(\begin{bmatrix} -17/40 \\ -1/10 \\ 7/8 \\ 0 \end{bmatrix}\right) \approx -1.138$

3. To show that

$$\begin{bmatrix} 0_{m\times m} & A \\ A^t & Q \end{bmatrix}^{-1} = \begin{bmatrix} S^{-1} & -S^{-1}AQ^{-1} \\ -Q^{-1}A^tS^{-1} & Q^{-1}(Q + A^tS^{-1}A)Q^{-1} \end{bmatrix}, \tag{2.2}$$

we will verify that the product of $\begin{bmatrix} 0_{m\times m} & A \\ A^t & Q \end{bmatrix}$ and $\begin{bmatrix} S^{-1} & -S^{-1}AQ^{-1} \\ -Q^{-1}A^tS^{-1} & Q^{-1}(Q + A^tS^{-1}A)Q^{-1} \end{bmatrix}$

is the $(m+n)$-by-$(m+n)$ identity matrix. Viewing each of these two partitioned matrices as being comprised of four blocks, we calculate four different products as follows:

Top left block of product:

$$\begin{aligned} 0_{m\times m}S^{-1} - AQ^{-1}A^tS^{-1} &= -AQ^{-1}A^t(-AQ^{-1}A^t)^{-1} \qquad (2.3)\\ &= I_{m\times m} \end{aligned}$$

Bottom left block of product:

$$\begin{aligned} A^tS^{-1} - QQ^{-1}A^tS^{-1} &= A^tS^{-1} - A^tS^{-1} \qquad (2.4)\\ &= 0_{n\times m} \qquad (2.5) \end{aligned}$$

Top right block of product:

$$\begin{aligned} -0_{m\times m}S^{-1}AQ^{-1} + AQ^{-1}\left(Q + A^tS^{-1}A\right)Q^{-1} &= AQ^{-1}\left(Q + A^tS^{-1}A\right)Q^{-1} \qquad (2.6)\\ &= AQ^{-1}QQ^{-1} + AQ^{-1}A^tS^{-1}AQ^{-1}\\ &= AQ^{-1} + AQ^{-1}A^t\left(-AQ^{-1}A^t\right)^{-1}AQ^{-1} \qquad (2.7)\\ &= AQ^{-1} - I_{m\times m}AQ^{-1} \qquad (2.8)\\ &= 0_{m\times n} \qquad (2.9) \end{aligned}$$

Bottom right block of product:

$$\begin{aligned} -A^tS^{-1}AQ^{-1} + QQ^{-1}\left(Q + A^tS^{-1}A\right)Q^{-1} &= -A^tS^{-1}AQ^{-1} + \left(Q + A^tS^{-1}A\right)Q^{-1} \qquad (2.10)\\ &= -A^tS^{-1}AQ^{-1} + I_{n\times n} + A^tS^{-1}AQ^{-1}\\ &= I_{n\times n}. \qquad (2.11) \end{aligned}$$

If we combine the results of (2.3)-(2.10), we see that

$$\begin{aligned} \begin{bmatrix} 0_{m\times m} & A \\ A^t & Q \end{bmatrix} \begin{bmatrix} S^{-1} & -S^{-1}AQ^{-1} \\ -Q^{-1}A^tS^{-1} & Q^{-1}(Q + A^tS^{-1}A)Q^{-1} \end{bmatrix} &= \begin{bmatrix} I_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix}\\ &= I_{m+n}. \end{aligned}$$

4. We seek to maximize the probability an individual is heterozygous, namely $2x_1x_2 + 2x_1x_3 + 2x_2x_3$, subject to the constraints that x_1, x_2, and x_3 are nonnegative and sum to 1. Equivalently, we seek to minimize $f(\mathbf{x}) = -(2x_1x_2 + 2x_1x_3 + 2x_2x_3)$ subject to these same constraints. Since

the Hessian of f is $\begin{bmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}$, our quadratic programming problem, in matrix form, is given by

$$\begin{aligned} \text{minimize } f(\mathbf{x}) &= \frac{1}{2}\mathbf{x}^t Q\mathbf{x} + \mathbf{p}^t\mathbf{x} \\ \text{subject to} & \\ A\mathbf{x} &= \mathbf{b} \\ C\mathbf{x} &\leq \mathbf{d}, \end{aligned}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $\mathbf{b} = 1$, $C = -I_3$, and $\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The Lagrangian has a unique KKT point at $\mathbf{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, with corresponding multiplier vectors, $\lambda_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mu_0 = \frac{4}{3}$, and corresponding objective value $f(\mathbf{x}_0) = -\frac{2}{3}$. None of the three inequality constraints are binding, so the bordered Hessian is simply

$$B \begin{bmatrix} 0 & 1 & 1 & 1 & \\ 1 & 0 & -2 & -2 & -2 \\ 1 & -2 & 0 & -2 & \\ 1 & -2 & -2 & 0 & \end{bmatrix}$$

In this case, $n = 3$, $m = 1$, and $k = 0$, so we only need to check that $\det(B)$ has the same sign as $(-1)^{m+k} = -1$. Since $\det(B) = -12$, $\mathbf{x}_0$ is optimal.

5. The solution of the bimatrix game is given by the solution of the quadratic programming problem

$$\min_{\mathbf{z},\mathbf{x},\mathbf{y}} f(\mathbf{z},\mathbf{x},\mathbf{y}) = \mathbf{e}^t\mathbf{z} - \mathbf{x}^t(A+B)\mathbf{y}$$

subject to

$$\begin{aligned} A^t\mathbf{y} &\le z_1\mathbf{e} \\ B\mathbf{x} &\le z_2\mathbf{e} \\ \mathbf{e}^t\mathbf{x} &= 1 \\ \mathbf{e}^t\mathbf{y} &= 1 \\ \mathbf{x},\mathbf{y} &\ge, \end{aligned}$$

If $B = -A$ and if we denote $\mathbf{z} = \begin{bmatrix} w \\ -z \end{bmatrix}$, then $\begin{bmatrix}\mathbf{z},\mathbf{x},\mathbf{y}\end{bmatrix}$ is feasible for the quadratic programming problem if and only if all 4 of the following conditions hold:

(a) $B\mathbf{x} \le -z$,i.e., $A\mathbf{x} \ge z$,

(b) $A^t\mathbf{y} \le w$, i.e., $\mathbf{y}^tA \le w$,

(c) $\mathbf{e}^t\mathbf{x} = 1$, and

(d) $\mathbf{e}^t\mathbf{y} = 1$.

But for the 2 LPs that comprise the zero-sum matrix game, these 4 conditions hold if and only if both $\mathbf{x}$ and $\mathbf{y}$ are primal and dual feasible, respectively, with corresponding objective values, w and z. Thus the solution of the quadratic programming problem coincides with the solutions of the LP and corresponding dual LP that comprise the solution of the zero-sum matrix game.

6. One equilibrium solution arises when Driver A always drives straight and Driver B always swerves. In this case, Driver A's pure strategy yields a payoff of 2 and Drivers B's a payoff of 0. An analogous result holds if the drivers' pure strategies are interchanged.

   To determine the mixed-strategy equilibrium, we set

$$Q = \begin{bmatrix} 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & -(A+B) \\ 0_{2\times2} & -(A+B)^t & 0_{2\times2} \end{bmatrix},$$

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and}$$

$$C = \begin{bmatrix} M_1 & 0_{2\times2} & A^t \\ M_2 & B & 0_{2\times2} \\ 0_{2\times2} & -I_2 & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & -I_2 \end{bmatrix},$$

where $M_1 = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$, and $M_2 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$. We also set $\mathbf{d} = \mathbf{0}_{8\times1}$, $E = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, and $\mathbf{b} = \mathbf{e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Using Maple's `QPSolve` command, we obtain a solution to the quadratic programming problem,

$$\begin{aligned} \text{minimize } f(\mathbf{w}) &= \frac{1}{2}\mathbf{w}^t Q\mathbf{w} + \mathbf{p}^t\mathbf{w} \\ \text{subject to} & \\ E\mathbf{w} &= \mathbf{b} \\ C\mathbf{w} &\leq \mathbf{d}, \end{aligned}$$

given by

$$\mathbf{w}_0 = \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{x}_0 \\ \mathbf{y}_0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

Thus Driver A's equilibrium strategy consists of $\mathbf{x}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, implying that he chooses to drive straight or to swerve with equal probabilities of $\frac{1}{2}$ and that his payoff is $z_{0,1} = \frac{1}{2}$. Driver B's strategy and payoff are exactly the same.

If we instead calculate the KKT points of the quadratic programming problem, we discover that there are exactly five:

(a) $\mathbf{w}_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, with corresponding Lagrange multiplier vectors, $\boldsymbol{\lambda}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

and $\boldsymbol{\mu}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and with objective value, $f(\mathbf{w}_0) = 0$

(b) $\mathbf{w}_0 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, with corresponding Lagrange multiplier vectors, $\boldsymbol{\lambda}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

and $\boldsymbol{\mu}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and with objective value, $f(\mathbf{w}_0) = 0$

(c) $\mathbf{w}_0 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$, with corresponding Lagrange multiplier vectors, $\boldsymbol{\lambda}_0 =$
$\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{\mu}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, and with objective value, $f(\mathbf{w}_0) = 0$

(d) $\mathbf{w}_0 = \begin{bmatrix} 1/4 \\ 5/4 \\ 1/4 \\ 3/4 \\ 3/4 \\ 1/4 \end{bmatrix}$, with corresponding Lagrange multiplier vectors, $\boldsymbol{\lambda}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{\mu}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and with objective value, $f(\mathbf{w}_0) = \frac{1}{4}$

(e) $\mathbf{w}_0 = \begin{bmatrix} 5/4 \\ 1/4 \\ 3/4 \\ 1/4 \\ 1/4 \\ 3/4 \end{bmatrix}$, with corresponding Lagrange multiplier vectors, $\boldsymbol{\lambda}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{\mu}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and with objective value, $f(\mathbf{w}_0) = \frac{1}{4}$

The first two KKT points correspond to the two pure strategy equilibria and the third KKT to the mixed strategy equilibrium. That these are solutions of the original quadratic programming problem can be verified by use of the bordered Hessian test. By mere inspection of objective values, we see that the latter two KKT points not solutions of the quadratic programming problem. In fact, the bordered Hessian test is inconclusive. Because these KKT points are not solutions of the problem, they do not constitute Nash equilibria.

Solutions to Waypoints